On classification of finite dimensional algebras

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Abstract

A constructive approach to the classification and invariance problems, with respect to basis changes, of the finite dimensional algebras is offered. A construction of an invariant open, dense (in the Zariski topology) subset of the space of structural constants of algebras and a classification of the corresponding algebras by providing a separating system of rational invariants are given. A finite system of generators for the corresponding field of invariant rational functions is shown.

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1 Introduction

The classification of finite dimensional algebras is an important problem in Algebra. For example, the classification of finite dimensional simple and semi-simple associative algebras by Wedderburn, the classification of finite dimensional simple and semi-simple Lie algebras by Cartan are well known. Their classifications are examples of structural (basis free, invariant) approaches to the classification problem of algebras. The structural approach becomes more difficult and unclear when one considers more general types of algebras. Another disadvantage of such approach is that the classification is assumed to be only with respect to the general linear group. In reality one may be interested also in classification of algebras with respect to specific changes of basis.

Another approach to the classification problem of algebras is coordinate (basis based, structural constants) approach. For the small dimensional cases for such approach one can see [1-3]. In general case the basis based approach is considered in [4]. We also consider the classification and invariants problem of finite dimensional algebras in general case. Though there are some intersecting results in this paper with those of [4] our used tools are more elementary and more constructive than of [4]. We note (Remark 2.1) that our approach is applicable to some classical subgroups of general linear group case as well.

In general for the given dimension we provide a method how to construct an invariant, open, dense subset of the space of structural constants of algebras and classify all algebras who's system of structural constants are in this set. We provide a basis for the field of invariant rational functions of structural constants as well.

The paper is organized in the following way. The key results which are used to obtain the classification and invariants of algebras are presented in Section 2. Section 3 can be considered as a realization of Section 2 results in the case of representation of general linear group in the space of structural constants of algebras.

2 Preliminaries

In this section we consider a linear representation of a subgroup of the general linear group and under an assumption prove some general results about the equivalence and invariance problems with respect to this subgroup.

Let n, m be any natural numbers, $\tau : (G, V) \to V$ be a fixed linear algebraic representation of an algebraic subgroup G of $GL(m, F)$ over V, where F is any field and V is n-dimensional vector space over F. Further we consider this representation under the following assumption:

Assumption. There exists a nonempty G-invariant subset V_0 of V and an algebraic map $P : V_0 \to G$ such that

$$P(\tau(g, \mathbf{v})) = P(\mathbf{v})g^{-1} \tag{1}$$

whenever $\mathbf{v} \in V_0$ and $g \in G$.

Theorem 2.1. Elements $\mathbf{u}, \mathbf{v} \in V_0$ are G-equivalent, that is $\mathbf{u} = \tau(g, \mathbf{v})$ for some $g \in G$, if and only if $\tau(P(\mathbf{u}), \mathbf{u}) = \tau(P(\mathbf{v}), \mathbf{v})$.

Proof. If $\mathbf{u} = \tau(g, \mathbf{v})$ then $\tau(P(\mathbf{u}), \mathbf{u}) = \tau(P(\tau(g, \mathbf{v})), \tau(g, \mathbf{v})) =$

$$\tau(P(\mathbf{v})g^{-1}, \tau(g, \mathbf{v})) = \tau(P(\mathbf{v}), \tau(g^{-1}, \tau(g, \mathbf{v}))) = \tau(P(\mathbf{v}), \mathbf{v}).$$

Visa versa, if $\tau(P(\mathbf{u}), \mathbf{u}) = \tau(P(\mathbf{v}), \mathbf{v})$ then

$$\tau(P(\mathbf{u})^{-1}P(\mathbf{v}), \mathbf{v}) = \tau((P(\mathbf{u}))^{-1}, \tau(P(\mathbf{v}), \mathbf{v})) = \tau((P(\mathbf{u}))^{-1}, \tau(P(\mathbf{u}), \mathbf{u})) = \mathbf{u}$$

that is $\mathbf{u} = \tau(g, \mathbf{v})$, where $g = P(\mathbf{u})^{-1}P(\mathbf{v})$.

This proposition shows that the system of components of $\tau(P(\mathbf{x}), \mathbf{x})$ is a separating system of invariants for the G-orbits in V_0, where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is an algebraic independent system of variables over F.

Further in this paper it is assumed that F is an algebraically closed field of characteristic zero, V_0 in the above Assumption is a dense (in Zariski topology) in V and G-invariant. In such case for any $\mathbf{u}, \mathbf{v} \in V_0$ one has $P(\tau(P(\mathbf{u}), \mathbf{v})) = P(\mathbf{v})P(\mathbf{u})^{-1}$ and due to density of V_0 in V one has

$$P(\tau(P(\mathbf{y}), \mathbf{x})) = P(\mathbf{x})P(\mathbf{y})^{-1},$$

where $\mathbf{y} = (y_1, y_2, ..., y_n)$ is also an algebraic independent system of variables over F.

Theorem 2.2. The field of G-invariant rational functions $F(\mathbf{x})^G$ is generated over F by the system of components of $\tau(P(\mathbf{x}), \mathbf{x})$.

Proof. It is evident that all components of $\tau(P(\mathbf{x}), \mathbf{x})$ are in $F(\mathbf{x})^G$. If $f(\mathbf{x}) = f(\tau(g, \mathbf{x}))$ for all $g \in G$ then, in particular, $f(\mathbf{x}) = f(\tau(P(\mathbf{u}), \mathbf{x}))$ whenever $\mathbf{u} \in V_0$. It implies, as far as V_0 is dense in V, that for the variable vector $\mathbf{y}$ the equality

$$f(\mathbf{x}) = f(\tau(P(\mathbf{y}), \mathbf{x}))$$

holds true. In $\mathbf{y} = \mathbf{x}$ case one gets that $f(\mathbf{x}) = f(\tau(P(\mathbf{x}), \mathbf{x}))$.

Corollary 2.1. The field $F(\mathbf{x})$ is generated over $F(\mathbf{x})^G$ by the system of components of $P(\mathbf{x})$.

Proof. Indeed $F(\mathbf{x})^G(P(\mathbf{x})) = F(\tau(P(\mathbf{x}), \mathbf{x}))(P(\mathbf{x})) = F(\tau(P(\mathbf{x}), \mathbf{x}), P(\mathbf{x}))$ and
$\tau(P(\mathbf{x})^{-1}, \tau(P(\mathbf{x}), \mathbf{x})) = \mathbf{x}$ and therefore $F(\mathbf{x})^G(P(\mathbf{x})) = F(\mathbf{x})$.

Proposition 2.1. The equality $\mathrm{trdeg}F(P(\mathbf{x}))/F = \mathrm{dim}G$ holds true.

Proof. To prove the equality it is enough to show equality of the vanishing ideals of $P(\mathbf{x})$ and G. If polynomial p vanished on $P(\mathbf{x})$, that is $p[P(\mathbf{x})] = 0$, then $p[P(\tau(g, \mathbf{x},))] = p[P(\mathbf{x})g^{-1}] = 0$. In particular, $p[g] = 0$ for any $g \in G$ that is p vanishes on G as well.

If $p[g] = 0$ for any $g \in G$ then, in particular, $p[P(\mathbf{u})] = 0$ for any $\mathbf{u} \in V_0$. Due to density of V_0 in V one has $p[P(\mathbf{x})] = 0$.

Theorem 2.3. The equality $\mathrm{trdeg}F(\mathbf{x})^G/F = n - \mathrm{dim}G$ holds true.

Proof. Let $\widetilde{P(\mathbf{x})}$ stand for any system of entries of $P(\mathbf{x})$ which is a transcendence basis for the field $F(P(\mathbf{x}))$ over F. We show that the system $\widetilde{P(\mathbf{x})}$ is algebraic independent over $F(\mathbf{x})^G$ as well. Indeed let $p[(y_{ij})_{i,j=1,2,...,m}]$ be any polynomial over $F(\mathbf{x})^G$ for which $p[\widetilde{P(\mathbf{x})}] = 0$ that is $p_{\mathbf{v}}[\widetilde{P(\mathbf{v})}] = 0$ for all $\mathbf{v} \in V_1$, where V_1 is a G-invariant nonempty open subset of V_0, where $p_{\mathbf{v}}[(y_{ij})_{i,j=1,2,...,m}]$ stands for the polynomial obtained from $p[(y_{ij})_{i,j=1,2,...,m}]$

by substitution $\mathbf{v}$ for $\mathbf{x}$. The equality $0 = p_{\mathbf{v}}[\widetilde{P(\mathbf{v})}] = p_{\tau(g,\mathbf{v})}[\widetilde{P(\tau(g,\mathbf{v}))}] = p_{\mathbf{v}}[\widetilde{P(\mathbf{v})g^{-1}}]$ implies that $p_{\mathbf{v}}[\widetilde{g}] = 0$ for any $g \in G$. Therefore $p_{\mathbf{v}}[\widetilde{P(\mathbf{x})}] = 0$, that is $p_{\mathbf{v}}[\widetilde{(y_{ij})}_{i,j=1,2,\ldots,m}]$ is zero polynomial for any $\mathbf{v} \in V_1$. It means that $p[\widetilde{(y_{ij})}_{i,j=1,2,\ldots,m}]$ is zero polynomial itself. Now due to $F \subset F(\mathbf{x})^G \subset F(\mathbf{x})$, tr.deg.$F(\mathbf{x})/F = n$ and Corollary 2.1 one has the required result.

For $G = GL(m,F)$ case due to Theorem 2.3 one has the following result.

Corollary 2.2 The transcendence degree of $F(\mathbf{x})^{GL(m,F)}$ over F equals to $n - m^2$ and the field extension $F(\mathbf{x})^{GL(m,F)} \subset F(\mathbf{x})$ is a pure transcendental extension. For a transcendental basis one can take the system of components of $P(\mathbf{x})$.

Remark 2.1 The above presented results show the importance of having for the given subgroup G the map $P(x)$ with the properties as in the Assumption. The following two statements are evident.

a) If for a subgroup G one has $P(x)$ with the needed properties then for the group $SG = G \cap SL(m,F)$ one can construct $P(x)$ with the needed properties by defining it to be $P(x) := \frac{1}{\det P(x)} P(x)$.

b) To get similar $P(x)$ for the orthogonal group $G = O(n,F)$, provided that one knows $P(x)$ for $GL(m,F)$, one can apply the Gram-Schmidt's orthogonal process to the system of rows of $P(x)$ if possible. As a result one gets an orthogonal matrix $Q(x)P(x)$, where $Q(x)$ is an upper triangular matrix with $O(m,F)$-invariant entries. For $O(m,F)$ one can consider this $Q(x)P(x)$ for it's $P(x)$.

Remark 2.2 The Assumption may be productive in a slightly different form as well: There exists a nonempty G-invariant subset V_0 of V and an algebraic map $P : V_0 \to GL(m,F)$ such that $P(\tau(g,\mathbf{v})) = P(\mathbf{v})g^{-1}$ whenever $\mathbf{v} \in V_0$ and $g \in G$. In this case Theorem 2.1 can be formulated in the following form: Elements $\mathbf{u},\mathbf{v} \in V_0$ are G-equivalent, that is $\mathbf{u} = \tau(g,\mathbf{v})$ for some $g \in G$, if and only if $\tau(P(\mathbf{u}),\mathbf{u}) = \tau(P(\mathbf{v}),\mathbf{v})$ and $P(\mathbf{u})^{-1}P(\mathbf{v}) \in G$. For the classical subgroups of $GL(m,F)$ the relation $P(\mathbf{u})^{-1}P(\mathbf{v}) \in G$ can be written in terms of equality of some G-invariants on $P(\mathbf{u})$ and $P(\mathbf{v})$, for example, in $G = O(m,F)$-the orthogonal group case it means $P(\mathbf{u})P(\mathbf{u})^t = P(\mathbf{v})P(\mathbf{v})^t$. So in such cases for the classical subgroups the separating system of invariants can be listed easily. But whether the separating system generates the corresponding field of invariant functions is unclear even one admits density of V_0 in V.

Question 2.1. Under the assumption for $G = GL(m, F)$ is it true that $F \subset F(\mathbf{x})^{GL(m,F)}$ is also a pure transcendental extension?

3 Classification of algebras

3.1 General case

In this paper we use the standard notation (the Einstein notation) for tensors as well as the matrix representation for tensors which is more convenient in dealing with equivalence and invariance problems of tensors with respect to basis changes. The use of matrix representation for tensors makes the descriptions more transparent as well. Further the classification problem is considered only with respect to the general linear group.

Let us consider any m dimensional algebra W with multiplication $\cdot$ given by a bilinear map $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$. If $e = (e^1, e^2, ..., e^m)$ is a basis for W then one can represent the bilinear map by a matrix $A \in Mat(m \times m^2; F)$ such that

$$\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v)$$

for any $\mathbf{u} = eu, \mathbf{v} = ev$, where $u = (u_1, u_2, ..., u_m), v = (v_1, v_2, ..., v_m)$ are column vectors. So the binary operation (bilinear map, tensor) is presented by the matrix $A \in Mat(m \times m^2; F)$ with respect to the basis e. Further we deal only with such matrices of rank m.

If $e' = (e'^1, e'^2, ..., e'^m)$ is also a basis for W, $g \in G = GL(m, F)$, $e'g = e$ and $\mathbf{u} \cdot \mathbf{v} = e'B(u' \otimes v')$, where $\mathbf{u} = e'u', \mathbf{v} = e'v'$, then $\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v) = e'B(u' \otimes v') = eg^{-1}B(gu \otimes gv) = eg^{-1}B(g \otimes g)(u \otimes v)$ as far as $\mathbf{u} = eu = e'u' = eg^{-1}u', \mathbf{v} = ev = e'v' = eg^{-1}v'$. Therefore the equality

$$B = gA(g^{-1})^{\otimes 2} \tag{2}$$

is valid.

Now let τ stand for the representation of $G = GL(m, F)$ on the $n = m^3$ dimensional vector space $V = Mat(m \times m^2; F)$ defined by

$$\tau : (g, A) \mapsto B = gA(g^{-1} \otimes g^{-1}).$$

To have Theorems 2.1-2.3 for this case we will construct a map $P : V_0 \to GL(m, F)$ with property (1) in the following way. For any natural number k due to (2) one has

$$B^{\otimes k} = g^{\otimes k} A^{\otimes k} (g^{-1})^{\otimes 2k} \tag{3}$$

Let us consider all its possible contractions with respect to k upper and k lower indices. It is clear that the result of each of such contraction will be $f(B) = f(A)(g^{-1})^{\otimes k}$ type equality, where $f(A)$ is a row vector with m^k entries.

In $k = 1$ case one gets the following $2^1 1! = 2$ different row equalities: $\mathbf{Tr_1}(B) = \mathbf{Tr_1}(A)g^{-1}$, $\mathbf{Tr_2}(B) = \mathbf{Tr_2}(A)g^{-1}$, where $\mathbf{Tr_1}(A)$ stands for the row vector with entries $A^j_{j,i} = \sum_{j=1}^n A^j_{j,i}$- the contraction on the first upper and lower indices and $\mathbf{Tr_2}(A)$ stands for the row vector with entries $A^j_{i,j} = \sum_{j=1}^n A^j_{i,j}$- the contraction on the first upper and second lower indices.

In $k = 2$ case one gets the following $2^2 2! + 2^1 1! = 10$ different row equalities:

$$\mathbf{Tr_i}(B)\otimes\mathbf{Tr_j}(B) = (\mathbf{Tr_i}(A)\otimes\mathbf{Tr_j}(A))(g^{-1})^{\otimes 2},\ \mathbf{Tr_i}(B)B = \mathbf{Tr_i}(A)A(g^{-1})^{\otimes 2},$$

where $i, j = 1, 2$, and

$$(B^i_{j,p}B^j_{i,q}) = (A^i_{j,p}A^j_{i,q})(g^{-1})^{\otimes 2},\ (B^i_{j,p}B^j_{q,i}) = (A^i_{j,p}A^j_{q,i})(g^{-1})^{\otimes 2},$$

$$(B^i_{p,j}B^j_{i,q}) = (A^i_{p,j}A^j_{i,q})(g^{-1})^{\otimes 2},\ (B^i_{p,j}B^j_{q,i}) = (A^i_{p,j}A^j_{q,i})(g^{-1})^{\otimes 2}.$$

In any k case only the number of contractions of $A^{\otimes k}$ when all k different upper indices are contracted with lower indices of different A is

$$(2k) \times (2(k-1)) \times (2(k-2)) \times ... \times 2 = 2^k k!.$$

In general it is nearly clear that the corresponding resulting system of $2^k k!$ rows depending on the variable matrix $A := \mathbf{x} = (x^i_{j,k})_{i,j,k=1,2,...,m}$ is linear independent over F. But for big enough k the inequality $2^k k! \geq m^k$ holds true as well. Therefore in general for big enough k it is possible to choose m^k contractions (rows) among the all contractions of $\mathbf{x}^{\otimes k}$ for which the matrix $Q(\mathbf{x})$ consisting of these m^k rows is a nonsingular matrix. For the matrix $Q(\mathbf{x})$ one has equality $Q(\mathbf{y}) = Q(\mathbf{x})(g^{-1})^{\otimes k}$ whenever $g \in G$, $\mathbf{y} = g\mathbf{x}(g^{-1})^{\otimes 2}$.

Now note that for any $A \in \{\mathbf{x} : \det(Q(\mathbf{x})) \neq 0\}$ and $g \in G$ one has, for example, $(B \otimes (\mathbf{Tr_1}(B))^{\otimes k-2})Q(B)^{-1} =$

$$g(A\otimes(\mathbf{Tr_1}(A))^{\otimes k-2})(g^{-1})^{\otimes k}(Q(A)(g^{-1})^{\otimes k})^{-1} = g(A\otimes(\mathbf{Tr_1}(A))^{\otimes k-2})Q(A)^{-1}.$$

Therefore if $P(A)^{-1}$ stands for arbitrary nonsingular $m\times m$ size sub-matrix of $(A \otimes (\mathbf{Tr_1}(A))^{\otimes k-2})Q(A)^{-1}$ then one has the equality $P(B)^{-1} = gP(A)^{-1}$, where $g \in G$, $B = gA(g^{-1})^{\otimes 2}$. It implies that whenever $A \in V_0 = \{A :$

$\det(P(A))\det(Q(A)) \neq 0\}$ the equality $P(B) = P(A)g^{-1}$ holds true for any $g \in G$ and $B = gA(g^{-1})^{\otimes 2}$. Note that

$$V_0 = \{A : \det(P(A))\det(Q(A)) \neq 0\}$$

is a G-invariant, open and dense subset of V.

Therefore we have the following results.

Theorem 3.1. Two algebras with matrices of structural constants $A, B \in V_0$ are same (isomorph) algebras if and only if

$$P(A)A(P(A)^{-1} \otimes P(A)^{-1}) = P(B)B(P(B)^{-1} \otimes P(B)^{-1}).$$

Theorem 3.2. The field of G-invariant rational functions $F(\mathbf{x})^G$ of structural constants defined by variable matrix $\mathbf{x} = (\mathbf{x}^i_{j,k})_{i,j,k=1,2,\ldots,m}$ is generated by the system of entries of $P(\mathbf{x})\mathbf{x}(P(\mathbf{x})^{-1} \otimes P(\mathbf{x})^{-1})$ over F, that is the equality

$$F(\mathbf{x})^G = F(P(\mathbf{x})\mathbf{x}(P(\mathbf{x})^{-1} \otimes P(\mathbf{x})^{-1}))$$

holds true.

Theorem 3.3. The transcendence degree of $F(\mathbf{x})^G$ over F equals to $m^3 - m^2$ and the field extension $F(\mathbf{x})^G \subset F(\mathbf{x})$ is a pure transcendental extension.

Remark 3.1. One of the main results (Theorem 1) of [4] states that the field extension $F \subset F(\mathbf{x})^{GL(m,F)}$ is a pure transcendental extension, which so far we could not get by our approach. Theorem 3.3 can be considered as a complementary result to that Theorem.

Now let us consider two and three dimensional algebra cases.

Example 3.l. Two dimensional ($m = 2$) case. Let

$$A = \begin{pmatrix} A^1_{1,1} & A^1_{1,2} & A^1_{2,1} & A^1_{2,2} \\ A^2_{1,1} & A^2_{1,2} & A^2_{2,1} & A^2_{2,2} \end{pmatrix}$$

be the matrix of structural constants with respect to a basis. In this case at $k = 1$ already $2^1 1! = m^1$ and therefore for the rows of $P(A)$ on can take

$$\mathbf{Tr_1}(A) = (A^1_{1,1} + A^2_{2,1}, A^1_{1,2} + A^2_{2,2}) \text{ and } \mathbf{Tr_2}(A) = (A^1_{1,1} + A^2_{1,2}, A^1_{2,1} + A^2_{2,2})$$

and V_0 consists of all A for which

$$\det P(A) = (A^1_{1,1} + A^2_{2,1})(A^1_{2,1} + A^2_{2,2}) - (A^1_{1,2} + A^2_{2,2})(A^1_{1,1} + A^2_{1,2}) \neq 0.$$

To see the corresponding system of generators one can evaluate $P(\mathbf{x})\mathbf{x}(P(\mathbf{x})^{-1} \otimes P(\mathbf{x})^{-1})$, where $\mathbf{x} = \begin{pmatrix} x^1_{1,1} & x^1_{1,2} & x^1_{2,1} & x^1_{2,2} \\ x^2_{1,1} & x^2_{1,2} & x^2_{2,1} & x^2_{2,2} \end{pmatrix}$.

On classification problem of two dimensional algebras one can see [1,2].

Example 3.2. Three dimensional ($m=3$) case. Let

$$A=\begin{pmatrix} A^1_{1,1} & A^1_{1,2} & A^1_{1,3} & A^1_{2,1} & A^1_{2,2} & A^1_{2,3} & A^1_{3,1} & A^1_{3,2} & A^1_{3,3}\\ A^2_{1,1} & A^2_{1,2} & A^2_{1,3} & A^2_{2,1} & A^2_{2,2} & A^2_{2,3} & A^2_{3,1} & A^2_{3,2} & A^2_{3,3}\\ A^3_{1,1} & A^3_{1,2} & A^3_{1,3} & A^3_{2,1} & A^3_{2,2} & A^3_{2,3} & A^3_{3,1} & A^3_{3,2} & A^3_{3,3}\end{pmatrix}$$

be the matrix of the structural constants with respect to a basis.

In this case at $k=1$ one has $2^1 1! < 3^1$. At $k=2$ already $2^2 2! + 2^1 1! = 10 > 3^2$ and the following 10 equalities

$$\mathbf{Tr_i}(B)\otimes \mathbf{Tr_j}(B)=\mathbf{Tr_i}(A)\otimes \mathbf{Tr_j}(A)(g^{-1})^{\otimes 2},\ \mathbf{Tr_i}(B)B=\mathbf{Tr_i}(A)A(g^{-1})^{\otimes 2},$$

where $i,j=1,2$,

$$(B^i_{j,p}B^j_{i,q})=(A^i_{j,p}A^j_{i,q})(g^{-1})^{\otimes 2},\ (B^i_{j,p}B^j_{q,i})=(A^i_{j,p}A^j_{q,i})(g^{-1})^{\otimes 2},$$

$$(B^i_{p,j}B^j_{i,q})=(A^i_{p,j}A^j_{i,q})(g^{-1})^{\otimes 2},\ (B^i_{p,j}B^j_{q,i})=(A^i_{p,j}A^j_{q,i})(g^{-1})^{\otimes 2}$$

hold true.

Therefore, for example, for $Q(A)$ one can take the following matrix

$$Q(A)=\left(\begin{array}{lllll} A^i_{i,1}A^j_{j,1} & A^i_{i,1}A^j_{j,2} & A^i_{i,1}A^j_{j,3} & A^i_{i,2}A^j_{j,1} & A^i_{i,2}A^j_{j,2}\\ A^i_{i,1}A^j_{1,j} & A^i_{i,1}A^j_{2,j} & A^i_{i,1}A^j_{3,j} & A^i_{i,2}A^j_{1,j} & A^i_{i,2}A^j_{2,j}\\ A^i_{1,i}A^j_{j,1} & A^i_{1,i}A^j_{j,2} & A^i_{1,i}A^j_{j,3} & A^i_{2,i}A^j_{j,1} & A^i_{2,i}A^j_{j,2}\\ A^i_{1,i}A^j_{1,j} & A^i_{1,i}A^j_{2,j} & A^i_{1,i}A^j_{3,j} & A^i_{2,i}A^j_{1,j} & A^i_{2,i}A^j_{2,j}\\ A^i_{i,j}A^j_{1,1} & A^i_{i,j}A^j_{1,2} & A^i_{i,j}A^j_{1,3} & A^i_{i,j}A^j_{2,1} & A^i_{i,j}A^j_{2,2}\\ A^i_{j,i}A^j_{1,1} & A^i_{j,i}A^j_{1,2} & A^i_{j,i}A^j_{1,3} & A^i_{j,i}A^j_{2,1} & A^i_{j,i}A^j_{2,2}\\ A^i_{j,1}A^j_{i,1} & A^i_{j,1}A^j_{i,2} & A^i_{j,1}A^j_{i,3} & A^i_{j,2}A^j_{i,1} & A^i_{j,2}A^j_{i,2}\\ A^i_{j,1}A^j_{1,i} & A^i_{j,1}A^j_{2,i} & A^i_{j,1}A^j_{3,i} & A^i_{j,2}A^j_{1,i} & A^i_{j,2}A^j_{2,i}\\ A^i_{1,j}A^j_{i,1} & A^i_{1,j}A^j_{i,2} & A^i_{1,j}A^j_{i,3} & A^i_{2,j}A^j_{i,1} & A^i_{2,j}A^j_{i,2} \end{array}\right.$$

$$\left.\begin{matrix}
A^i_{i,2}A^j_{j,3} & A^i_{i,3}A^j_{j,1} & A^i_{i,3}A^j_{j,2} & A^i_{i,3}A^j_{j,3} \\
A^i_{i,2}A^j_{3,j} & A^i_{i,3}A^j_{1,j} & A^i_{i,3}A^j_{2,j} & A^i_{i,3}A^j_{3,j} \\
A^i_{2,i}A^j_{j,3} & A^i_{3,i}A^j_{j,1} & A^i_{3,i}A^j_{j,2} & A^i_{3,i}A^j_{j,1} \\
A^i_{2,i}A^j_{3,j} & A^i_{3,i}A^j_{1,j} & A^i_{3,i}A^j_{2,j} & A^i_{3,i}A^j_{3,j} \\
A^i_{i,j}A^j_{2,3} & A^i_{i,j}A^j_{3,1} & A^i_{i,j}A^j_{3,2} & A^i_{i,j}A^j_{3,3} \\
A^i_{j,i}A^j_{2,3} & A^i_{j,i}A^j_{3,1} & A^i_{j,i}A^j_{3,2} & A^i_{j,i}A^j_{3,3} \\
A^i_{j,2}A^j_{i,3} & A^i_{j,3}A^j_{i,1} & A^i_{j,3}A^j_{i,2} & A^i_{j,3}A^j_{i,3} \\
A^i_{j,2}A^j_{3,i} & A^i_{j,3}A^j_{1,i} & A^i_{j,3}A^j_{2,i} & A^i_{j,3}A^j_{3,i} \\
A^i_{2,j}A^j_{i,3} & A^i_{3,j}A^j_{i,1} & A^i_{3,j}A^j_{i,2} & A^i_{3,j}A^j_{i,3}
\end{matrix}\right).$$

For $P(A)^{-1}$ one can take any 3×3 size nonsingular sub-matrix of $(A \otimes \mathbf{Tr_1}(A))Q(A)^{-1}$, where $(A \otimes \mathbf{Tr_1}(A)) =$

$$\left(\begin{matrix}
A^1_{1,1}A^i_{i,1} & A^1_{1,1}A^i_{i,2} & A^1_{1,1}A^i_{i,3} & A^1_{1,2}A^i_{i,1} & A^1_{1,2}A^i_{i,2} \\
A^2_{1,1}A^i_{i,1} & A^2_{1,1}A^i_{i,2} & A^2_{1,1}A^i_{i,3} & A^2_{1,2}A^i_{i,1} & A^2_{1,2}A^i_{i,2} \\
A^3_{1,1}A^i_{i,1} & A^3_{1,1}A^i_{i,2} & A^3_{1,1}A^i_{i,3} & A^3_{1,2}A^i_{i,1} & A^3_{1,2}A^i_{i,2}
\end{matrix}\right.$$

$$\left.\begin{matrix}
A^1_{1,2}A^i_{i,3} & A^1_{1,3}A^i_{i,1} & A^1_{1,3}A^i_{i,2} & A^1_{1,3}A^i_{i,3} \\
A^2_{1,2}A^i_{i,3} & A^2_{1,3}A^i_{i,1} & A^2_{1,3}A^i_{i,2} & A^2_{1,3}A^i_{i,3} \\
A^3_{1,3}A^i_{i,3} & A^3_{1,3}A^i_{i,1} & A^3_{1,3}A^i_{i,2} & A^3_{1,3}A^i_{i,3}
\end{matrix}\right).$$

3.2 Commutative and anti-commutative algebra cases

For the classification purpose instead of all m dimensional algebras one can consider only such commutative or anti-commutative algebras. The commutativity (anti-commutativity) of the binary operation in terms of the corresponding matrix A means $A^i_{j,k} = A^i_{k,j}$ (respectively, $A^i_{j,k} = -A^i_{k,j}$) for all $i, j, k = 1, 2, ..., m$. So in commutative (anti-commutative) algebra case for the V we consider $V =$

$$\{A \in Mat(m \times m^2; F) : \ A^i_{j,k} = A^i_{k,j} (\text{resp. } A^i_{j,k} = -A^i_{k,j}) \text{ for all } i, j, k = 1, 2, ..., m.\}$$

Note that in commutative (anti-commutative) case the dimension of V is $\frac{m^2(m+1)}{2}$ (respectively, $\frac{m^2(m-1)}{2}$).

To have Theorems 2.1-2.3 for these cases one can construct a map $P : V_0 \to GL(m, F)$ with property (1) in a similar way as in the general algebra case. Consider once again equality (3) and all its possible contractions with respect to k upper and k lower indices.

In commutative (anti-commutative) case at $k = 1$ one gets the following $1! = 1$ row equality: $\mathbf{Tr_1}(B) = \mathbf{Tr_1}(A)g^{-1} = \mathbf{Tr_2}(A)g^{-1}$ as far as $A^i_{j,k} = A^i_{k,j}$ (respectively, $\mathbf{Tr_1}(B) = \mathbf{Tr_1}(A)g^{-1} = -\mathbf{Tr_2}(A)g^{-1}$ as far as $A^i_{j,k} = -A^i_{k,j}$) for all $i, j, k = 1, 2, ..., m$.

In $k = 2$ case one gets the following $2! + 1! = 3$ different row equalities:

$$\mathbf{Tr_1}(B) \otimes \mathbf{Tr_1}(B) = (\mathbf{Tr_1}(A) \otimes \mathbf{Tr_1}(A))(g^{-1})^{\otimes 2},$$

$$\mathbf{Tr_1}(B)B = \mathbf{Tr_1}(A)A(g^{-1})^{\otimes 2},\ (B^i_{j,p}B^j_{i,q}) = (A^i_{j,p}A^j_{i,q})(g^{-1})^{\otimes 2}.$$

In any k case only the number of contractions of $A^{\otimes k}$ when all k different upper indices are contracted with lower indices of different A is $k!$. Once again in general it is nearly clear that the corresponding resulting system of $k!$ rows depending on variable matrix $A := \mathbf{x} = (x^i_{j,k})_{i,j,k=1,2,...,m}$, where $x^i_{j,k} = x^i_{k,j}$ (respectively, $x^i_{j,k} = -x^i_{k,j}$) for all $i, j, k = 1, 2, ..., m$, is linear independent over F. But for big enough k the inequality $k! \geq m^k$ holds true as well. Therefore in general for big enough k it is possible to choose m^k contractions (rows) among the all contractions of $\mathbf{x}^{\otimes k}$ for which the matrix $Q(\mathbf{x})$ consisting of these m^k rows is nonsingular. For the matrix $Q(\mathbf{x})$ one has equality $Q(\mathbf{y}) = Q(\mathbf{x})(g^{-1})^{\otimes k}$ whenever $g \in G$, $\mathbf{y} = g\mathbf{x}(g^{-1})^{\otimes 2}$.

Therefore if $P(A)^{-1}$ stands for arbitrary $m \times m$-size nonsingular submatrix of $(A \otimes (\mathbf{Tr_1}(A))^{\otimes k-2})Q(A)^{-1}$ then one has the equality $P(B)^{-1} = gP(A)^{-1}$, where $g \in G$, $B = gA(g^{-1})^{\otimes 2}$. It implies that whenever $A \in V_0 = \{A \in V : \det(P(A))\det(Q(A)) \neq 0\}$ the equality $P(B) = P(A)g^{-1}$ holds true for any $g \in G$, where $B = gA(g^{-1})^{\otimes 2}$. Note that

$$V_0 = \{A \in V : \det(P(A))\det(Q(A)) \neq 0\}$$

is a G-invariant, open and dense subset of V.

Therefore we have the following results.

Theorem 3.1'. Two commutative (anti-commutative) algebras with the matrices of structural constants $A, B \in V_0$ are the same algebras if and only if

$$P(A)A(P(A)^{-1} \otimes P(A)^{-1}) = P(B)B(P(B)^{-1} \otimes P(B)^{-1}).$$

Theorem 3.2'. The field of G-invariant rational functions $F(\mathbf{x})^G$ of the structural constants presented by the matrix $\mathbf{x} = ((x^i_{j,k})_{i,j,k=1,2,...,m}$ of the variable commutative (respectively, anti-commutative) algebras, where $x^i_{j,k} = x^i_{k,j}$ (respectively, $x^i_{j,k} = -x^i_{k,j}$) for all $i, j, k = 1, 2, ..., m$, is generated by the system of entries of $P(\mathbf{x})\mathbf{x}(P(\mathbf{x})^{-1} \otimes P(\mathbf{x})^{-1})$ over F, that is the equality

$$F(\mathbf{x})^G = F(P(\mathbf{x})\mathbf{x}(P(\mathbf{x})^{-1} \otimes P(\mathbf{x})^{-1}))$$

holds true.

Theorem 3.3'. In commutative (anti-commutative) algebra case the transcendence degree of $F(\mathbf{x})^G$ over F equals to $\frac{m^2(m-1)}{2}$ (respectively, $\frac{m^2(m-3)}{2}$, $m \geq 3$) and the field extension $F(\mathbf{x})^G \subset F(\mathbf{x})$ is a pure transcendental extension.

Now let us consider two dimensional commutative algebra case.

Example 3.1'. Let

$$A = \begin{pmatrix} A^1_{1,1} & A^1_{1,2} & A^1_{2,1} & A^1_{2,2} \\ A^2_{1,1} & A^2_{1,2} & A^2_{2,1} & A^2_{2,2} \end{pmatrix}$$

, where $A^i_{1,2} = A^i_{2,1}$ at $i = 1, 2$, be the matrix of structural constants of a commutative algebra with respect to a basis. Consider $B^{\otimes 3} = g^{\otimes 3} A (g^{-1})^{\otimes 6}$ and its all contractions on 3 upper and 3 lower indices. Among them in particular one gets the following 6 equalities:

$$(B^i_{\sigma(i),p} B^j_{\sigma(j),q} B^k_{\sigma(k),r}) = (A^i_{\sigma(i),p} A^j_{\sigma(j),q} A^k_{\sigma(k),r})(g^{-1})^{\otimes 3}$$

, where $\sigma \in S_3$- the symmetric group of permutations of symbols i, j, k, and

$(B^i_{p,q} B^j_{r,k} B^k_{i,j}) = (A^i_{p,q} A^j_{r,j} A^k_{i,j})(g^{-1})^{\otimes 3}$, $(B^i_{p,q} B^j_{r,i} B^k_{k,j}) = (A^i_{p,q} A^j_{r,i} A^k_{k,j})(g^{-1})^{\otimes 3}$.

So for $Q(A)$ one can take the matrix consisting of the following 8 rows

$$(A^i_{\sigma(i),p} A^j_{\sigma(j),q} A^k_{\sigma(k),r})_{\sigma \in S_3}, (A^i_{p,q} A^j_{r,k} A^k_{i,j}), (A^i_{p,q} A^j_{r,i} A^k_{k,j})$$

and for the $P(A)^{-1}$ any nonsingular 2×2 -size sub-matrix of $(A \otimes Tr_1(A))Q(A)^{-1}$ provided that $\det(Q(A)) \neq 0$.

On classification of three dimensional anti-commutative algebras one can see [3].

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