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On Central Extensions of algebras

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Motivation

Let us start with some introductory considerations on Lie algebra extensions. Loosely speaking an extension of a Lie algebra M is an enlargement of M by some other Lie algebra. Starting with two Lie algebras K and M over the same field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ we consider the Cartesian product $L := M \times K$. Elements of this set are ordered pairs (m, k) with $m \in M$ and $k \in K$. Defining addition of such pairs by $(m, k) + (m', k') := (m + m', k + k')$ and multiplication by scalars $a \in \mathbb{F}$ as $a(m, k) := (am, ak)$ the set L becomes a vector space. Using the Lie brackets on M and K we define on L the bracket

$$[(m, k), (m', k')] := ([m, m'], [k, k']).$$

Motivation

One easily sees that this is a Lie bracket on L. Defining next the map $\alpha : k \in K \longmapsto (0, k) \in L$ and the map $\beta : (m, k) \in L \longmapsto m \in M$ one readily verifies that α is an injective Lie algebra homomorphism, while β is a surjective Lie algebra homomorphism. Moreover, $im\alpha = ker\beta$. The Lie algebra L with these properties is called *a trivial extension* of M by K (or of K by M). Instead of denoting the elements of L by ordered pairs we will frequently use the notation $(m, k) = m + k$. The vector space L is then written as $L = M \oplus K$ and the Lie bracket is in this notation given by

$$[m + k, m' + k']_L = [m, m']_M + [k, k']_K.$$

Motivation

This example of a trivial extension is generalized in the following concept. Let K, L and M be Lie algebras and let these algebras be related in the following way.

- There exists an injective Lie algebra homomorphism
$$\alpha : K \longrightarrow L.$$
- There exists an surjective Lie algebra homomorphism
$$\beta : L \longrightarrow M.$$
- The Lie algebra homomorphisms α and β are related by
$$im\ \alpha = ker\ \beta.$$

Then the Lie algebra L is called *an extension* of M by K. The relationship is summarized by the sequence

$$K \xrightarrow{\alpha} L \xrightarrow{\beta} M.$$

Motivation

One obtains that $ker\ \beta$ is an ideal in L and since β is surjective we have a Lie algebra isomorphism between the quotient algebra $L/ker\ \beta$ and M,

$$L/ker\ \beta \cong M.$$

Using $im\ \alpha = ker\ \beta$ this relation can be written as

$$L/im\ \alpha \cong M.$$

Since α is injective the Lie algebras K and $im\ \alpha$ are Lie isomorphic. From these properties one sees that it makes sense to call the Lie algebra L is an extension of M by K.

Motivation

Proposition

The sequence $K \longrightarrow L \longrightarrow M$ of Lie algebras is an extension of M by K if and only if the sequence

$$0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} M \longrightarrow 0$$

is exact.

Motivation

Definition

An extension $K \xrightarrow{\alpha} L \xrightarrow{\beta} M$ is called:

- ***trivial*** *if there exists an ideal $I \subset L$ complementary to ker β, i.e.*

$$L = \text{ker } \beta \oplus I \quad \text{(Lie algebra direct sum)},$$

- ***split*** *if there exists a Lie subalgebra $S \subset L$ complementary to ker β, i.e.*

$$L = \text{ker } \beta \oplus S \quad \text{(vector space direct sum)},$$

- ***central*** *if the ker β is contained in the center $Z(L)$ of L, i.e. ker $\beta \subset Z(L)$.*

Extension by a derivation

Let L be a Lie algebra and $\delta : L \longrightarrow L$ be a derivation of L. Define a Lie bracket on the vector space direct sum $L \oplus \mathbb{F}\delta$ by

$$[x + \lambda\delta, y + \mu\delta] := [x, y]_L + \lambda\delta(y) - \mu\delta(x),$$

where $x, y \in L$, $\lambda, \mu \in \mathbb{F}$ and $[\cdot, \cdot]_L$ is the Lie bracket on L. The vector space $L \oplus \mathbb{F}\delta$ equipped with this Lie bracket is a Lie algebra which is denoted by L_δ. Notice that the right-hand side of the equation above is always an element of L, and consequently L is an ideal in L_δ. Furthermore, $\mathbb{F}\delta$ is a (one-dimensional abelian) subalgebra of L_δ and the vector space $\mathbb{F}\delta$ is complementary to L.

Extension by a derivation

Elements in $L_\delta = L \oplus \mathbb{F}\delta$ have the form $(x, \mu\delta) = x + \mu\delta (x \in L, \mu \in \mathbb{F})$. Defining α and β by $\alpha : x \in L \longmapsto (x, 0) \in L_\delta$ and $\beta : (x, \mu\delta) \in L_\delta \longmapsto \mu\delta \in \mathbb{F}\delta$ one sees that α is injective while β is surjective. Furthermore, *im* $\alpha =$ *ker* β and hence the sequence

$$L \xrightarrow{\alpha} L_\delta \xrightarrow{\beta} \mathbb{F}\delta$$

is an extension of L by $K = \mathbb{F}\delta$. Since $L_\delta =$ *ker* $\beta \oplus \mathbb{F}\delta$ with *ker* α is an ideal and $\mathbb{F}\delta$ is a subalgebra, we have a split extension

$$L_\delta = L \oplus \mathbb{F}\delta,$$

which is called an *extension by a derivation*.

Lie algebras

The polynomial loop algebra associated to a finite-dimensional Lie algebra is given as follows.

Definition

Let L be a Lie algebra. Elements of the polynomial loop algebra $\tilde{L}$ are linear combinations of elements of the form $P(\lambda)x$ with $x \in L$ and $P(t)$ a Laurent polynomial in the indeterminate t. The commutation relations are fixed by

$$[\lambda^m x, \lambda^n y]_{\tilde{L}} = \lambda^{m+n}[x, y]_L \quad m, n \in \mathbb{Z}. \tag{1}$$

Lie algebras

- Loop algebras;
- Loop algebra of a simple Lie algebra = Kac-Moody algebra;
- Witt algebra $\mathcal{W}$: - Lie algebra of vector fields on the unit cycle S^1;
- Virasoro algebra $\mathcal{V}$: it is an extension of $\mathcal{W}$ and is used to establish the boson-fermion correspondence.

Central Extensions of Lie algebras

So far central extensions of groups and algebras mostly are extensively studied in Lie algebras case and the results successfully applied in various branches of physics. In the theory of Lie groups, Lie algebras and their representation theory, a Lie algebra extension M is an enlargement of a given Lie algebra L by another Lie algebra K. Extensions arise in several ways. There is the trivial extension obtained by taking a direct sum of two Lie algebras. Other types are the split extension and the central extension. Extensions may arise naturally, for instance, when forming a Lie algebra from projective group representations.

Applications of CE

It is proven that a finite-dimensional simple Lie algebra has only trivial central extensions. Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. Kac-Moody algebras have been conjectured to be a symmetry groups of a unified superstring theory. The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in M-theory.

Applications

The central extensions also are applied in the process of pre-quantization, namely in the construction of prequantum bundles in geometric quantization.
Starting with a polynomial loop algebra over finite-dimensional simple Lie algebra and performing two extensions, a central extension and an extension by a derivation, one obtains a Lie algebra which is isomorphic with an untwisted affine Kac-Moody algebra. Using the centrally extended loop algebra one may construct a current algebra in two spacetime dimensions. The Virasoro algebra is the universal central extension of the Witt algebra.

Group Extensions

Due to the Lie correspondence, the theory, and consequently the history of Lie algebra extensions, is tightly linked to the theory and history of group extensions. A systematic study of group extensions has been given by the Austrian mathematician O. Schreier in 1923. Lie algebra extensions are most interesting and useful for infinite-dimensional Lie algebras. In 1967, Victor Kac and Robert Moody independently generalized the notion of classical Lie algebras, resulting in a new theory of infinite-dimensional Lie algebras, now called Kac-Moody algebras. They generalize the finite-dimensional simple Lie algebras and can often concretely be constructed as extensions.

Classes of algebras

- Associative algebras A;
- Lie algebras $\mathfrak{g}$;
- Pre-Lie algebras g;
- Leibniz algebras L;
- Zinbiel algebras R;
- Associative Dialgebras D.

2-Cocycles for algebras

Let $\mathfrak{A}$ be an algebra and V be a vector space. A bilinear function $\theta : \mathfrak{A} \times \mathfrak{A} \longrightarrow V$ is said to be

- an associative 2-cocycle of $\mathfrak{A} = A$ with values in V if

$$\theta(ab, c) = \theta(a, bc) \text{ for all } a, b, c \in A.$$

- a Lie 2-cocycle of $\mathfrak{A} = \mathfrak{g}$ with values in V if

$$\theta(x, y) = -\theta(y, x$$

$$\theta([x, y], z) + \theta([y, z], x) + \theta([z, x], y) = 0,$$

  for all $x, y, z \in \mathfrak{g}$.

- a Pre-Lie 2-cocycle of $\mathfrak{A} = \mathrm{g}$ with values in V if

$$\theta(xy, z) - \theta(x, yz) = \theta(xz, y) - \theta(x, zy),$$

  for all $x, y, z \in \mathrm{g}$.

Cocycles for algebras

- a Leibniz 2-cocycle of $\mathfrak{A} = L$ with values in V if

$$\theta([x, y], z) = \theta([x, z], y) + \theta(x, [y, z]),$$

  for all $x, y, z \in L$.
- a Zenbiel 2-cocycle of $\mathfrak{A} = R$ with values in V if

$$\theta(x \circ y, z) = \theta(x, z \circ y) + \theta(x, y \circ z),$$

  for all $x, y, z \in R$.

Cohomology Groups for algebras

The set of all 2-cocycles on algebra $\mathfrak{A}$ with values in V is denoted by $Z^2(\mathfrak{A}, V)$.
Let $\nu : \mathfrak{A} \longrightarrow \mathfrak{A}$ be a linear map. Define $\eta(a, b) = \nu(a * b)$.
Then η is a 2-cocycle called coboundary. The set of all 2-coboundaries on $\mathfrak{A}$ with values in V is denoted by $B^2(\mathfrak{A}, V)$ and $H^2(\mathfrak{A}, V)$ is called the second group of cohomologies with values in V.

Algebras defined by cocycles

Theorem

Let $(\mathfrak{A}, \lambda)$ be an algebra, (where $\mathfrak{A} = A$, $\mathfrak{g}$, L or R,) and V a vector space, $\theta : \mathfrak{A} \times \mathfrak{A} \longrightarrow V$ be bilinear map. Put $\mathfrak{A}_\theta = \mathfrak{A} \oplus V$. For $x, y \in \mathfrak{A}$ and $v, w \in V$ we define

$$\lambda_\theta((x+v),(y+w)) = \lambda(x,y) + \theta(x,y).$$

Then $(\mathfrak{A}, \lambda_\theta)$ is an associative, Lie, Pre-Lie, Leibniz or Zinbiel algebra if and only if θ is the corresponding associative, Lie, Leibniz, Pre-Lie or Zinbiel 2-cocycle, respectively.

Associative dialgebras

A dissociative algebra is a vector space with two bilinear binary associative operations $\vdash$, $\dashv$, satisfying certain conditions. Associative algebras are particular case of the diassociative algebras where the two operations coincide. The class of the associative dialgebras has been introduced by Loday in 1990 (see [2] and references therein). The main motivation of Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic K-theory. Later some their important relations with classical and non-commutative geometry, and physics have been discovered.

Dialgebra axioms

Definition

Associative dialgebra (the term "diassociative algebra" also is used) D over a field $\mathbb{F}$ is an algebra equipped with two bilinear binary associative operations $\dashv$ and $\vdash$, called left and right products, respectively, satisfying the following axioms:

$$\begin{aligned} (x \dashv y) \dashv z &= x \dashv (y \vdash z), \\ (x \vdash y) \dashv z &= x \vdash (y \dashv z), \\ (x \dashv y) \vdash z &= x \vdash (y \vdash z), \end{aligned} \tag{2}$$

for all $x, y, z \in D$.

Category of Associative dialgebras

A homomorphism from associative dialgebra $(D_1, \dashv_1, \vdash_1)$ to associative dialgebra
$(D_2, \dashv_2, \vdash_2)$ is a linear map $f : D_1 \longrightarrow D_2$ satisfying

$$f(x \dashv_1 y) = f(x) \dashv_2 f(y) \text{ and } f(x \vdash_1 y) = f(x) \vdash_2 f(y),$$

for all $x, y \in D_1$.
The kernel and the image of a homomorphism is defined naturally. A bijective homomorphism is called isomorphism. Main problem in structural theory of algebras is the problem of classification. The classification means the description of the orbits under base change linear transformations and list representatives of the orbits.

Center. Automorphism Group

The center of D is defined by

$$Z(D) = \{x \in D \mid x * y = y * x = 0 \text{ for all } y \in D\},$$

where $* = \dashv$ and $\vdash$.
The natural extensions of the concepts of nilpotency and solvability of algebras to diassociative algebras case have been given in [1]. The automorphism group of D is denoted by $AutD$.

Approaches applied

There is the following naive approach for classifying of algebra structures on a vector space which often is being used up to now. It runs as follows. Let us fix a basis of the underlying vector space, then according to the identities which the algebra satisfies we get a system of equations with respect to the structure constants of the algebra on this basis. Solving this system of equations we get a redundant, in general, list of algebras via the tables of multiplications. The second step is to make the obtained list irredundant.

Classification Results

The irredundancy can be achieved by identifying those algebras which are obtained from others by a base change. This approach has been applied to get classifications of two and three-dimensional diassociative algebras over $\mathbb{C}$. The classification results can be found in [2], [3], [7] in low-dimensional cases and four-dimensional nilpotent case has been obtained in [5]. In this paper we propose a different approach to the classification problem of associative dialgebras based on central extensions and action of group automorphisms on the Grassmanian of subspaces of the second cohomological groups of algebra with a smaller dimension.

Central Extensions of Associative dialgebras

Now we introduce the concept of central extension of associative dialgebras.

Definition

Let D_1, D_2 and D_3 be associative dialgebras over a field $\mathbb{F}$. The associative dialgebra D_2 is called the **extension** *of D_3 by D_1 if there are associative dialgebra homomorphisms $\alpha : D_1 \longrightarrow D_2$ and $\beta : D_2 \longrightarrow D_3$ such that the following sequence*

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0$$

is exact.

Equivalence of CE

Definition

An extension is called ***trivial*** *if there exists an ideal I of D_2 complementary to ker β, i.e.,*

$$D_2 = ker\ \beta \oplus I \quad \textit{(the direct sum of algebras).}$$

Definition

Two sequences

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0$$

and

$$0 \longrightarrow D_1 \xrightarrow{\alpha'} D_2' \xrightarrow{\beta'} D_3 \longrightarrow 0$$

are called ***equivalent extensions*** *if there exists a associative dialgebra isomorphism $f : D_2 \longrightarrow D_2'$ such that*

$$f \circ \alpha = \alpha' \text{ and } \beta' \circ f = \beta.$$

Central Extensions

The equivalence of extensions is an equivalent relation.

Definition

An extension

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0 \tag{3}$$

is called ***central*** *if the kernel of β is contained in the center $Z(D_2)$ of D_2, i.e., ker $\beta \subset Z(D_2)$.*

From the definition above it is easy to see that in a central extension the algebra D_1 must be abelian. A central extension of an associative dialgebra D_2 by an abelian associative dialgebra D_1 can be obtained with the help of 2-cocycles on D_3.

Cocycles on Diassociative algebras

This section introduces the concept of 2-cocycle for diassociative algebras and gives some of simple but important properties of the 2-cocycles.

Definition

*Let D be an associative dialgebra over a field $\mathbb{F}$ and V be a vector space over the same field. A pair $\Theta = (\theta_1, \theta_2)$ of bilinear maps $\theta_1 : D \times D \longrightarrow V$ and $\theta_2 : D \times D \longrightarrow V$ is called a 2-**cocycle** on D with values in V if θ_1 and θ_2 satisfy the conditions*

$$\begin{aligned}
\theta_1(x \dashv y, z) &= \theta_1(x, y \dashv z) &&= \theta_1(x, y \vdash z),\\
\theta_2(x \vdash y, z) &= \theta_2(x, y \vdash z) &&= \theta_2(x \dashv y, z),\\
\theta_1(x \vdash y, z) &= \theta_2(x, y \dashv z),
\end{aligned} \tag{4}$$

for all $x, y, z \in D$.

Coboundaries

The set of all 2-cocycles on D with values in V is denoted by $Z^2(D,V)$. One easily sees that $Z^2(D,V)$ is a vector space if one defines the vector space operations as follows:

$$(\Theta_1 \oplus \Theta_2)(x,y) := \Theta_1(x,y) + \Theta_2(x,y) \text{ and } (\lambda \odot \Theta)(x) := \lambda\Theta(x).$$

It is easy to see that a linear combination of 2-cocycles again is a 2-cocycle. A special types of 2-cocycles given by the following lemma are called 2-**coboundaries**.

Lemma

Let $\nu : D \longrightarrow V$ be a linear map, and define $\varphi_1(x,y) = \nu(x \dashv y)$ and $\varphi_2(x,y) = \nu(x \vdash y)$. Then $\Phi = (\varphi_1, \varphi_2)$ is a cocycle.

Construction of Coboundaries

Proof.

Let us check the axioms (4) one by one.

$$\begin{aligned} \varphi_1(x \dashv y, z) &= \nu((x \dashv y) \dashv z) \\ &= \nu(x \dashv (y \dashv z)) \quad = \varphi_1(x, y \dashv z). \end{aligned}$$

$$\begin{aligned} \varphi_1(x, y \dashv z) &= \nu(x \dashv (y \dashv z)) \\ &= \nu(x \dashv (y \vdash z)) \quad = \varphi_1(x, y \vdash z). \end{aligned}$$

$$\begin{aligned} \varphi_2(x \vdash y, z) &= \nu((x \vdash y) \vdash z) \\ &= \nu(x \vdash (y \vdash z)) \quad = \varphi_2(x, y \vdash z). \end{aligned}$$

$\square$

Construction of Coboundaries

Proof.

$$\begin{aligned}\varphi_2(x, y \vdash z) &= \nu(x \vdash (y \vdash z)) \\ &= \nu((x \vdash y) \vdash z) \\ &= \nu((x \dashv y) \vdash z) \quad = \varphi_2(x \dashv y, z).\end{aligned}$$

$$\begin{aligned}\varphi_1(x \vdash y, z) &= \nu((x \vdash y) \dashv z) \\ &= \nu(x \vdash (y \dashv z)) = \varphi_2(x, y \dashv z).\end{aligned}$$

$\square$

Cohomology Group

The set of all coboundaries is denoted by $B^2(D,V)$. Clearly, $B^2(D,V)$ is a subgroup of $Z^2(D,V)$. The group $H^2(D,V) = Z^2(D,V)/B^2(D,V)$ is said to be a second cohomology group of D with values in V. Two cocycles Θ_1 and Θ_2 are said to be **cohomologous** cocycles if $\Theta_1 - \Theta_2$ is a coboundary. If we view V as a trivial D-bimodule, then $H^2(D,V)$ is an analogue of the second Hochschild-cohomology space.

Associative dialgebras defined by cocycles

Theorem

Let D be an associative dialgebra, V a vector space,

$$\theta_1 : D \times D \longrightarrow V \text{ and } \theta_2 : D \times D \longrightarrow V$$

be bilinear maps. Set $D_\Theta = D \oplus V$, where $\Theta = (\theta_1, \theta_2)$. For $x, y \in D$, $v, w \in V$ we define

$$(x + v) \dashv (y + w) = x \dashv y + \theta_1(x, y)$$

and

$$(x + v) \vdash (y + w) = x \vdash y + \theta_2(x, y).$$

Then $(D_\Theta, \dashv, \vdash)$ is an associative dialgebra if and only if $\Theta = (\theta_1, \theta_2)$ is a 2-cocycle.

Proof

Proof.

Let D_Θ be the diassociative defined above. We show that the functions θ_1, θ_2 are 2-cocycles. Indeed, according to axioms (2) we have

$$((x+v) \dashv (y+w)) \dashv (z+t) = (x+v) \dashv ((y+w) \dashv (z+t))$$

$$= (x+v) \dashv ((y+w) \vdash (z+t)).$$

$$((x+v) \vdash (y+w)) \vdash (z+t) = (x+v) \vdash ((y+w) \vdash (z+t))$$

$$= ((x+v) \dashv (y+w)) \vdash (z+t)).$$

$$((x+v) \vdash (y+w)) \dashv (z+t)) = (x+v) \vdash ((y+w)) \dashv (z+t)).$$

$\square$

Cont.

Proof.

Since

$$\begin{aligned}((x+v)\dashv(y+w))\dashv(z+t) &= ((x\dashv y)+\theta_1(x,y))\dashv(z+t)\\ &= ((x\dashv y)\dashv z)+\theta_1(x\dashv y,z).\end{aligned}$$

$$\begin{aligned}(x+v)\dashv((y+w))\dashv(z+t)) &= (x+v)\dashv((y\dashv z)+\theta_1(y,z))\\ &= (x\dashv(y\dashv z))+\theta_1(x,y\dashv z).\end{aligned}$$

$$\begin{aligned}(x+v)\dashv((y+w)\vdash(z+t)) &= (x+v)\dashv((y\vdash z)+\theta_2(y,z)\\ &= (x\dashv(y\vdash z))+\theta_1(x,y\vdash z).\end{aligned}$$

Comparing these equations we get

$$\theta_1(x\dashv y,z)=\theta_1(x,y\dashv z)=\theta_1(x,y\vdash z).$$

Cont.

Proof.

$$\begin{aligned}((x+v)\vdash(y+w))\vdash(z+t) &= ((x\vdash y)+\theta_2(x,y))\vdash(z+t)\\ &= ((x\vdash y)\vdash z)+\theta_2(x\vdash y,z).\end{aligned}$$

$$\begin{aligned}(x+v)\vdash((y+w))\vdash(z+t)) &= (x+v)\vdash((y\vdash z)+\theta_2(y,z))\\ &= (x\vdash(y\vdash z))+\theta_2(x,y\vdash z).\end{aligned}$$

$$\begin{aligned}((x+v)\dashv(y+w))\vdash(z+t) &= (x\dashv y+\theta_1(x,y))\vdash(z+t)\\ &= (x\dashv y)\vdash z)+\theta_2(x\dashv y,z).\end{aligned}$$

Comparing these equations we obtain

$$\theta_2(x\vdash y,z)=\theta_2(x,y\vdash z)=\theta_2(x\dashv y,z).$$

Cont.

Proof.

$$\begin{aligned}((x+v)\vdash(y+w))\dashv(z+t)) &= (x\vdash y+\theta_2(x,y)\dashv(z+t)\\ &= ((x\vdash y)\dashv z))+\theta_1(x\vdash y,z).\end{aligned}$$

$$\begin{aligned}(x+v)\vdash((y+w))\dashv(z+t)) &= (x+v)\vdash((y\dashv z)+\theta_1(y,z))\\ &= (x\vdash(y\dashv z)+\theta_2(x,y\dashv z).\end{aligned}$$

Comparing the later two we derive

$$\theta_1(x\vdash y,z)=\theta_2(x,y\dashv z).$$

$\square$

CE defined by coboundaries

Lemma

Let Θ be a cocycle and Φ a coboundary. Then $D_\Theta \cong D_{\Theta+\Phi}$.

Proof.

The isomorphism

$$f : D_\Theta \longrightarrow D_{\Theta+\Phi},$$

is given by $f(x+v) = x + \nu(x) + v$. First of all we note that f is a bijective linear transformation. The linear transformation f obeys the algebraic operations $\dashv$ and $\vdash$ as well. $\square$

Proof

Proof.

Indeed,

$$\begin{aligned} f((x+v)\dashv(y+w)) &= f(x\dashv y+\theta_1(x,y)) \\ &= f(x\dashv y)+f(\theta_1(x,y)) \\ &= x\dashv y+\nu(x\dashv y)+\theta_1(x,y) \\ &= x\dashv y+\varphi_1(x,y)+\theta_1(x,y) \\ &= x\dashv y+(\varphi_1+\theta_1)(x,y) \end{aligned}$$

$$\begin{aligned} = ((x+(\varphi_1+\theta_1)(x)+v))\dashv(y+(\varphi_1+\theta_1)(y)+w) \\ = f(x+v)\dashv f(y+w). \end{aligned}$$

For $\vdash$ the proof is carried out similarly. $\square$

AD defined by coboundaries

The proof of the following corollary is straightforward.

Corollary

Let $\Theta_1, \ \Theta_2$ be cohomologous 2-cocycles on a diassociative algebra D and $D_1, \ D_2$ be the central extensions constructed with these 2-cocycles, respectively. Then the central extensions D_1 and D_2 are equivalent extensions. Particularly, a central extension defined by a coboundary is equivalent with a trivial central extension.

Identifications

Let V be a k-dimensional vector space with a basis $e_1, e_2, \ldots, e_k$ and $\Theta = (\theta_1, \theta_2) \in Z^2(D, V)$. Then we have $Z^2(D, V) \cong Z^2(D, \mathbb{F})^k$, $B^2(D, V) \cong B^2(D, \mathbb{F})^k$ and

$$\Theta(x, y) = (\theta_1(x, y), \theta_2(x, y)) = \left(\sum_{i=1}^{k} \theta_1^i(x, y) e_i, \sum_{i=1}^{k} \theta_2^i(x, y) e_i \right).$$

Lemma

$\Theta = (\theta_1, \theta_2) \in Z^2(D, V)$ if and only if $\Theta^i = (\theta_1^i, \theta_2^i) \in Z^2(D, \mathbb{F})$.

Proof.

The proof is the direct verification of axioms (4). $\square$

An one to one correspondence

There is a close relationship between the central extensions and cocycles which is established in this section.

Theorem

There exists one to one correspondence between elements of $H^2(D, V)$ and nonequivalents central extensions of the diassociative algebra D by V.

Proof.

The fact that for a 2-cocycle Θ the algebra D_Θ is a central extension of D has been proven early.
Let us prove the converse, i.e., suppose that D_Θ is a central extension of D :

$$0 \longrightarrow V \xrightarrow{\alpha} D_\Theta \xrightarrow{\beta} D \longrightarrow 0.$$

□

Proof.

We construct a 2-cocycle which generates this central extension. Considering the extension with the property $im\ \alpha = ker\ \beta \subset Z(D)$ we show that one can find bilinear maps

$$\theta_1 : D_2 \times D_2 \longrightarrow D_1 \text{ and } \theta_2 : D_2 \times D_2 \longrightarrow D_1$$

which satisfy (4) for all $x, y, z \in D$ $(V = D_1)$.
In case D_1 is a one-dimensional associative dialgebra it can be identified with $\mathbb{F}$. To obtain maps satisfying equations (4) we consider a linear map $s : D_3 \longrightarrow D_2$ satisfying $\beta \circ s = id_{D_3}$. A map with this property is called a *section* of D_2. □

Bilinear maps

Proof.

With the help of a section one can define a bilinear maps $\vartheta_i : D_3 \times D_3 \longrightarrow D_2$ $(i = 1, 2)$ by taking for all $x, y \in D_2$

$$\vartheta_1(x, y) := s(x \dashv y) - s(x) \dashv s(y), \tag{5}$$

$$\vartheta_2(x, y) := s(x \vdash y) - s(x) \vdash s(y). \tag{6}$$

Notice that ϑ_i, $i = 1, 2$ is identically zero if s is an associative dialgebra homomorphism. Moreover, since β is an associative dialgebra homomorphism one sees that $\beta \circ \vartheta_i = 0$, $i = 1, 2$. Hence, for all $x, y \in D_3$ we have

$$\vartheta_i(x, y) \in ker\ \beta \subset Z(D_2). \tag{7}$$

□

Required cocycle

Proof.

Using this property and the associative dialgebra axioms one readily verifies that $\Omega = (\vartheta_1, \vartheta_2)$ satisfies

$$\begin{aligned} \vartheta_1(x \dashv y, z) &= \vartheta_1(x, y \dashv z) &= \vartheta_1(x, y \vdash z), \\ \vartheta_2(x \vdash y, z) &= \vartheta_2(x, y \vdash z) &= \vartheta_2(x \dashv y, z), \\ \vartheta_1(x \vdash y, z) &= \vartheta_2(x, y \dashv z). \end{aligned} \tag{8}$$

In the final step to obtain the 2-cocycle $\Theta = (\theta_1, \theta_2)$ we use the injectivity of the map $\alpha : D_1 \longrightarrow D_2$. The map $\Theta = (\theta_1, \theta_2)$, where $\theta_i : D_3 \times D_3 \longrightarrow D_1, \ i = 1, 2$ are defined by $\theta_i := \alpha^{-1} \circ \vartheta_i, \ i = 1, 2$, is the required cocycle. □

Action of $Aut\ D$ on $H^2(D, V)$

The algebra D_Θ is a (dim V-dimensional) central extension of D by V since we have the exact sequence of associative dialgebras

$$0 \longrightarrow V \longrightarrow D_\Theta \longrightarrow D \longrightarrow 0.$$

Let us now transfer the description of the orbits with respect to the "transport of structure" action of $GL(D_\Theta)$ ($dim\ D_\Theta = n$) on the variety of dialgebras $Dias_n$ to the description of orbits of action of the $AutD$ on the Grassmanian of $H^2(D, \mathbb{F})$. The action of $\sigma \in AutD$ on $\Theta \in Z^2(D, V)$ is defined as follows:

$$\sigma(\Theta(x,y)) = (\sigma(\theta_1(x,y)), \sigma(\theta_2(x,y)))\,,$$

where $\sigma(\theta_i(x,y)) = \theta_i(\sigma(x), \sigma(y)),\ i = 1, 2,$ for $x, y \in D$. Thus $AutD$ operates on $Z^2(D, V)$. It is easy to see that $B^2(D, V)$ is stabilized by this action, so that there is an induced action of $AutD$ on $H^2(D, V)$.

Action transformation

Theorem

Let

$$\Theta(x,y) = (\theta_1(x,y), \theta_2(x,y)) = \left(\sum_{i=1}^{k} \theta_1^i(x,y)e_i, \sum_{i=1}^{k} \theta_2^i(x,y)e_i \right)$$

and

$$\Omega(x,y) = (\omega_1(x,y), \omega_2(x,y)) = \left(\sum_{i=1}^{k} \omega_1^i(x,y)e_i, \sum_{i=1}^{k} \omega_2^i(x,y)e_i \right).$$

Then $D_\Theta \cong D_\Omega$ if and only if there exists $\varphi \in Aut(D)$ such that $\varphi(\omega_j^i)$ span the same subspace of $H^2(D,\mathbb{F})$ as the θ_j^i, $j = 1,2$ and $i = 1,2,...,k$.

Proof

Proof.

As vector spaces $D_\Theta = D \oplus V$ and $D_\Omega = D \oplus V$. Let $\sigma : D_\Theta \longrightarrow D_\Omega$ be an isomorphism. Since V is the center of both dialgebras, we have $\sigma(V) = V$. So σ induces an isomorphism of $D_\Theta/V = D$ to $D_\Omega/V = D$, i.e., it generates an automorphism φ of D. Let $D = Span\{x_1, ..., x_n\}$. Then we write $\sigma(x_i) = \varphi(x_i) + v_i$, where $v_i \in V$, and $\sigma(e_i) = \sum\limits_{j=1}^{s} a_{ji} e_j$.

Also write

$$x_i \dashv x_j = \sum_{k=1}^{n} \gamma_{ij}^k x_k, \quad x_i \vdash x_j = \sum_{k=1}^{n} \delta_{ij}^k x_k,$$

and $v_i = \sum\limits_{l=1}^{s} \beta_{il}^l e_l$. □

Cont.

Proof.

Then the relations

$$\sigma(x_i \dashv_\Theta x_j) = \sigma(x_i) \dashv_\Omega \sigma(x_j) \text{ and } \sigma(x_i \vdash_\Theta x_j) = \sigma(x_i) \vdash_\Omega \sigma(x_j)$$

amount to

$$\omega_r^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{lk}\theta_r^k(x_i, x_j) + \sum_{k=1}^{n} \gamma_{ij}^k \beta_{kl}, \text{ for } 1 \leq l \leq s$$

and

$$\omega_r^l(\varphi(x_i), \varphi(x_j)) = \sum_{k=1}^{s} a_{lk}\theta_r^k(x_i, x_j) + \sum_{k=1}^{n} \delta_{ij}^k \beta_{kl}, \text{ for } 1 \leq l \leq s,$$

where $r = 1, 2$. □

Cont.

Proof.

Now define the linear function $f_l : D \longrightarrow \mathbb{F}$ by $f_l(x_k) = \beta_{kl}$. Then

$$f_l^1(x_i \dashv x_j]) = \sum_{k=1}^{n} \gamma_{ij}^k \beta_{kl} \text{ and } f_l^2(x_i \vdash x_j]) = \sum_{k=1}^{n} \delta_{ij}^k \beta_{kl}.$$

From (9) and (9) it is obvious that modulo $B^2(D, \mathbb{F})$, $\varphi(\omega_r^i)$ and θ_r^i span the same space.
Let now suppose that there exists an automorphism $\varphi \in D$ such that the cocycles $\varphi(\omega_j^i)$ and θ_j^i, $j = 1, 2$ and $i = 1, 2, ..., k$ span the same space in $Z^2(D, \mathbb{F})$, modulo $B^2(D, \mathbb{F})$. Then there are linear functions $f_l : D \longrightarrow \mathbb{F}$ and $a_{lk} \in \mathbb{F}$ so that □

Cont.

Proof.

$$\omega_r^l(\varphi(x_i),\varphi(x_j)) = \sum_{k=1}^{s} a_{lk}\theta_r^k(x_i,x_j) + f_l(x_i \dashv x_j) \text{ for } 1 \leq l \leq s,$$
$$\omega_r^l(\varphi(x_i),\varphi(x_j)) = \sum_{k=1}^{s} a_{lk}\theta_r^k(x_i,x_j) + f_l(x_i \vdash x_j) \text{ for } 1 \leq l \leq s,$$

where $r = 1, 2$.
If we take $\beta_{kl} = f(x_k)$ then (9) and (9) hold. Now define $\sigma : D_\Theta \longrightarrow D_\Omega$ as follows

$$\sigma(x_i) = \varphi(x_i) + \sum_{l=1}^{s} \beta_{il}^l e_l, \quad \sigma(e_i) = \sum_{j=1}^{s} a_{ji} e_j.$$

Then the σ is the required isomorphism. □

Procedure

Let D' be an associative dialgebra and $Z(D')$ be its center which we suppose to be nonzero. Set $V = Z(D')$ and $D = D'/Z(D')$. Then there is a $\Theta \in H^2(D, V)$ such that $D' = D_\Theta$. We conclude that any associative dialgebra with a nontrivial center can be obtained as a central extension of an associative dialgebra of smaller dimension. So in particular, all nilpotent dialgebras can be constructed by this way.

Procedure: Let D be an associative dialgebra of dimension $n-s$. The procedure outputs all nilpotent algebras D' of dimension n such that $D'/Z(D') = D$. It runs as follows:

(1) Determine $Z^2(D, \mathbb{F})$, $B^2(D, \mathbb{F})$ and $H^2(D, \mathbb{F})$.

(2) Determine the orbits of $Aut(D)$ on s-dimensional subspaces of $H^2(D, \mathbb{F})$.

(3) For each of the orbits let Θ be the cocycle corresponding to a representative of it, and construct D_Θ.

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