INTRODUCTION TO LEFT SYMMETRIC DIALGEBRAS

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ABSTRACT

The research can be considered as an introduction to the structural theory of Left-Symmetric Dialgebras (LSD). This class of algebras has been introduced as a generalization of Left Symmetric Algebras (LSA). The structural theory of LSA is well developed, and there are many classification results in the literature. However, structural problems of LSD are still remain untouched. In the paper we do first effort to study structure theory of Left Symmetric Dialgebras: namely some notions have been introduced and two dimensional complex algebras are described. Maple program "Check LSD" will be introduced and applied.

Key words: *Left Symmetric algebra and dialgebra, isomorphism, automorphism, derivation, Rota-Baxter operator.*

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INTRODUCTION

Left Symmetric Algebras (LSA in short) have been introduced and studied by several mathematicians using different names, such as Pre-Lie algebras, Vinberg algebras, Koszul algebras or Quazi associative algebras see (Chapaton, 2001), (Caley, 1890), (Vinberg, 1963). There are some applications of LSA in geometry and physics have been explored by Burde (Burde, 1998). Importance of LSA is that not only application in geometry and physics but also they are closely related to Lie and Leibniz algebras (Burde, 2006). It is well developed structure theory of Lie algebras established in last century and most of results have been extended for Leibniz algebras, which is non-antisymmetric extension of Lie algebras (Loday, 1993). Both Lie and Leibniz algebras play important role in modern algebra, and there are some applications of such algebras in other branches of mathematics and physics. There exists a large literature on structural theory of Lie and Leibniz algebras.

Definition 1 Let A be a vector space over field F with a bilinear product $(x, y) \to xy$. A is called a associative algebra if, $\forall x, y, z \in A$

$$(x\,y)z = x(y\,z)$$

Definition 2 Let A be a vector space over field F with a bilinear product $(x, y) \to xy$. A is called a Lie algebra if $\forall x, y, z \in A$

$$[x, y] = -[y, x], \qquad \big[[x, y], z\big] + \big[[z, x], y\big] + \big[[y, z], x\big] = 0$$

- Relationship between two classical algebras such as Associative and Lie algebras can be acquired as follows: $[x, y] = xy - yx$ so we have $As \to Lie$.

Definition 3 (A, [,]) Leibniz algebra

$$\big[[x, y], z\big] = \big[[x, z], y\big] + \big[x, [y, z]\big]$$

If the bracket happens to be anti-commutative $[x, y] = -[y, x]$ then Leibniz algebra becomes Lie algebra. In other words Leibniz algebras are generalization of Lie algebras: $Lie \to Leib$

$$\begin{tikzcd} & \boxed{\text{Leib}} \\ \boxed{\text{As}} \arrow[r] & \boxed{\text{Lie}} \arrow[u, dotted] \end{tikzcd}$$

Definition 4 An associative dialgebra (or diassociative algebra) over a field F is a vector space D equipped with two bilinear associative binary operations denoted by $\dashv$ and $\vdash$ respectively, satisfying the following identities: $\forall x, y, z \in D$

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z).$$

If those two products of associative dialgebra are coincide then it becomes an associative algebra, in other words associative dialgebra is generalization of associative one, and it is valued the following implication:

$$As \rightarrow Dias: \quad x \cdot y = x \dashv y = x \vdash y$$

From Associative Dialgebra, (by similar procedure as above) can be obtained a Leibniz algebra ([LF]):

$$Dias \rightarrow Leib: \quad [x, y] = x \dashv y - y \vdash x,$$

$$\begin{tikzcd}
\boxed{\text{Dias}} \arrow[r] & \boxed{\text{Leib}} \\
\boxed{\text{As}} \arrow[r] \arrow[u, dotted] & \boxed{\text{Lie}} \arrow[u, dotted]
\end{tikzcd}$$

Definition 5 An algebra $(A, \cdot)$ over F with bilinear product $(x, y) \to x \cdot y$ is called Left-Symmetric Algebra (LSA), if product is left symmetric $(x, y, z) = (y, x, z)$

in other words, $\forall x, y, z \in A$: $\quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z)$

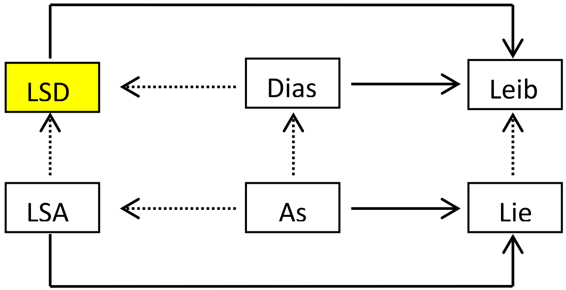
$$\begin{tikzcd}
 & \text{Dias} \arrow[r] & \text{Leib} \\
\text{LSA} & \text{As} \arrow[l, dotted] \arrow[u, dotted] \arrow[r] & \text{Lie} \arrow[u, dotted]
\end{tikzcd}$$

Definition 6 (R. Felipe, 2011) Let S be a vector space over a field F. Let us assume that S is equipped with two bilinear products, not necessary associative

i. $\quad x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
ii. $\quad (x \vdash y) \dashv z = x \vdash (y \dashv z)$
iii. $\quad x \dashv (y \dashv z) - (x \dashv y) \dashv z = y \vdash (x \dashv z) - (y \vdash x) \dashv z$
iv. $\quad x \vdash (y \vdash z) - (x \vdash y) \vdash z = y \vdash (x \vdash z) - (y \vdash x) \vdash z$

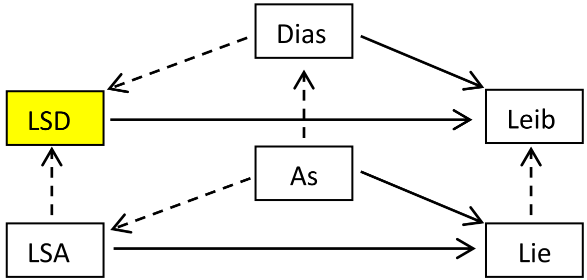
$\forall x, y, z \in S$ then $(S, \dashv, \vdash)$ is said to be a Left-Symmetric Dialgebra (Left di-symmetric algebra).
Note that LSD are generalization of Left Symmetric Algebras, i.e. we have the following implication:

$$LSA \to LSD: \quad x \cdot y = x \dashv y = x \vdash y$$



Including Left symmetric algebras and dialgebras "Loday diagram" (Loday, 2001) can be extended as shown in Figure above.

$$Lie \to Leib: \quad [x, y] = -[y, x]$$
$$Dias \to LSD: \quad (x \dashv y) \dashv z = x \dashv (y \dashv z), \qquad (x \vdash y) \vdash z = x \vdash (y \vdash z)$$
$$As \to LSA: \quad (xy)z = (yx)z$$
$$LSD \to Leib: \quad [x, y] = x \dashv y - y \vdash x$$



A classification of any class of algebras (even Associative algebras) over some field is a fundamental and one of the difficult problems of modern algebra. So far in Lie, Leibniz and Associative algebras cases the algebraic classification has been obtained up to certain dimensions. And note that by increasing dimension of algebra classification problems would be more challenging in terms of complexity of calculations, hence it is appropriate to consider some subclasses, such as nilpotent, filiform, quasi-filiform algebras.

In this paper we do first effort to investigate the structural theory of Left Symmetric Dialgebras (LSDs in short), which is generalization of LSA and closely related to Leibniz algebras. In section 2 Left Symmetric Algebras and Left-Symmetric Dialgebras (including classification of two dimensional complex LSD) are presented. Maple program is attached, which able to check whether obtained algebra is LSD or not.

Further all the algebras considered are supposed to be over the field F (assuming F is field of complex numbers).

LEFT-SYMMETRIC ALGEBRAS (LSA)

Let $(A, \cdot)$ be an algebra over field F, not necessarily associative and not necessarily finite dimensional. The associator (x, y, z) of three elements $x, y, z \in A$ is defined by

$$(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{1}$$

Left symmetric algebras – **E.B. Vinberg (1963)**, Such a notion was used in many studies related to geometry.

Pre-Lie algebra – due to close relation to Lie algebras, **Gerstenhaber (1963)** in study of deformations and cohomology theory of associative algebras.

Vinberg algebras - or Koszul algebra, Koszul-Vinberg algebra, whose notions are due to the pioneer work **J-L. Koszul (1961)** and **E.B. Vinberg (1963),** in study of affine manifolds and affine structure of Lie groups.

Quazi-associative algebra – Pre-Lie algebras (Left symmetric algebras) include associative algebras whose associators are zero. So in this sense, LSA can be regarded as kind of generalization of associative algebras, **Kupershmidt (1994)**.

Example 1.

Let $(A, \cdot)$ be a commutative associative algebra, and d be a derivation of A. Then the new product

$$x * y = x \cdot d(y),$$

$\forall x, y \in A$, makes $(A,*)$ become a left symmetric algebra.

Example 2.

Let V be a vector space over field $F$with the usual product (.,.) and let a be a fixed vector in A, then

$$x * y = (x, y)a + (x, a)y,$$

$\forall x, y \in V$, defines a left-symmetric algebra on V.

Example 3.

Let $(A, \cdot)$ be an associative algebra and $R: A \to A$ be a linear map satisfying

$$R(x) \cdot R(y) + R(x \cdot y) = R(R(x) \cdot y + x \cdot R(y))$$

$(\forall x, y \in A)$ defines a left-symmetric algebra on A. The linear map R is called Rota-Baxter map of the weight 1.

Note the opposite of Left Symmetric algebras are "Right Symmetric algebras", algebra with identity:

$$(x, y, z) = (x, z\, y)$$

The classification of LSA in low dimensions.

- The classification of 2-dimensional complex LSA was given in (C.Bai, 1996) and (D.Burde, 1998). The method is basically the computation of structure constants.
- The classification of 3-dimensional complex LSA was given in (C. Bai, 2009). It depends on a detailed study of 1-cocycles which divides the corresponding classification problem into solving a series of linear problems. It includes the classification of 3-dimensional complex Novikov algebras (C.Bai, 2001), bi-symmetric algebras (C.Bai, 2000) and simple LSA (D.Burde, 1998), which have been obtained independently.
- The classification of 3-dimensional real LSA was given in (X.Kong, .., 2012). It depends on the study of the relationships between real and complex LSA.
- The classification of 4-dimensional complex transitive LSA on nilpotent Lie algebras was given in (H. Kim, 1986). The method is to use an extension theory of LSA.
- The classification of 4-dimensional complex transitive simple LSA was given in (D.Burde, 1998).
- There are also some related classification results, like the classification of 3-dimensional LSA superalgebras and 2|2-dimensional Balinsky-Novikov superalgebras.

Remark: There are some results on classification of infinite dimensional LSA and Free LSA.

LEFT-SYMMETRIC DIALGEBRAS (LSD)

Example 4.

All dialgebras are Left-Symmetric Dialgebras, hence $D \subseteq S$.

Example 5

If $(A, \ \cdot)$ is a differential Left Symmetric Algebra, then the formula

$$x \dashv y = x \cdot d(y), \qquad x \vdash y = d(x) \cdot y$$

define a structure of Left Symmetric Dialgebra on A.

Example 6

Let K[x,y] be the polynomial algebra over a field K of characteristic 0. If we define two multiplications on K[x,y] as follows

$$f(x,y) \dashv g(x,y) = f(x,y)\, g(y,y)$$

$$f(x,y) \vdash g(x,y) = f(x,x)\, g(x,y)$$

then $(K[x,y], \dashv, \vdash)$ is a Left-Symmetric Dialgebra.

Note that in (L. Lin, 2010) shown that K[x,y] is associative dialgebra and Leibniz algebra.

Let S be a LSD algebra and M, N be subsets of S. We define

$$M \Diamond N = M \dashv N + M \vdash N$$

$$M \dashv N = Span\{x \dashv y \mid x \in M, y \in N\}$$

$$M \vdash N = Span\{x \vdash y \mid x \in M, y \in N\}$$

We are willing to introduce notion of nilpotency for LSD, as same way as we did for associative dialgebras, (Rikhsiboev, 2010). It is obvious that if M is left (N is right) ideal in S so is $M \Diamond N$ respectively. Therefore, if both M and N are two-sided ideals so is $M \Diamond N$.

Let us consider the following series of two-sided ideals:

$$S^1 = S,\ S^{k+1} = S^1 \Diamond S^k + S^2 \Diamond S^{k-1} + \cdots + S^k \Diamond S^1$$

Definition 7 A Left-Symmetric dialgebra S is said to be nilpotent if there exists natural number t such that $S^t = 0$.

An ideal I of S is said to be nilpotent if it is nilpotent as a subalgebra of S.

It is observed that the sum $I_1 + I_2 = \{z \in D | z = x_1 + x_2, x_1 \in I_1, x_2 \in I_2\}$ of two nilpotent ideals I_1, I_2 of S is nilpotent.

Therefore there exists unique maximal nilpotent ideal of S called nilradical. The nilradical plays an important role in the classification problem of algebras.

Proposition 1 A Left Symmetric Dialgebra S is an associative dialgebra if only if both products of S are associative.

Theorem 2 (R.Felipe, 2011) Let $(S, \dashv, \vdash)$ be a Left Symmetric Dialgebra. Then the following commutator

$$[x, y] = x \dashv y - y \vdash x$$

defines a structure of Leibniz algebra on S.

Notion of bar unit in Left Symmetric Dialgebras can be introduced similar way as done for associative dialgebras in (Loday, 2001).

CLASSIFICATION OF TWO DIMENSIONAL LSD

Since a LSD algebra possess two binary operations there are two right $R_x,\ r_x$ and two left $L_x,\ l_x$ multiplication operators defined as follows

$$R_x(y) := y \dashv x, \quad r_x(y) := y \vdash x, \quad L_x(y) := x \dashv y, \quad l_x(y) := x \vdash y$$

Lemma 3. For the left multiplication operators of LSD the following identities hold:

$$L_x L_y = L_x l_y, \quad l_{l_x(y)} = l_{L_x(y)},$$

$$L_x L_y - l_y L_x = L_{L_x(y)} - L_{l_y(x),}$$

$$l_x l_y - l_y l_x = l_{l_x(y)} - l_{l_y(x)}.$$

Definition 8 The sets defined by

$$Ann_R^{\dashv} = \{x \in S \mid S \dashv x = 0\}, \qquad Ann_R^{\vdash} = \{x \in S \mid Sx = 0\},$$

$$Ann_L^{\dashv} = \{x \in S \mid x \dashv S = 0\}, \qquad Ann_L^{\vdash} = \{x \in S \mid x \vdash S = 0\},$$

of a Left Symmetric Dialgebra S are called the right and the left annihilators of S, respectively.

Lemma 4 The right and left annihilators are two-sided ideals of S.

Definition 9 The annihilator of Left Symmetric Dialgebra S is $Ann(S) = Ann_R^{\dashv}(S) \cap Ann_L^{\vdash}(S)$.

Let S be an n-dimensional complex Left Symmetric Dialgebra and $e_1, e_2, \dots, e_n$ be its basis. If the components of basis products $e_i \dashv e_j$ and $e_i \vdash e_j$ where i, j=1,2,...,n, are defined as follows:

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k, \qquad e_i \vdash e_j == \sum_{k=1}^{n} \delta_{ij}^{k} e_k, \tag{2}$$

then the set $\{\gamma_{ij}^{k}, \delta_{ij}^{k} \in F, 1 \leq i, j, k \leq n\}$ is called the set of structure constants of S. This means that each point $\{\gamma_{ij}^{k},\ \delta_{ij}^{k}\}$ of the affine space K^{2n^2} defines an LSD structure on underlying vector space, however, for this structure to be a LSD structure the scalars $\{\gamma_{ij}^{k},\ \delta_{ij}^{k}\}$ must satisfy conditions according to the axioms of the LSD algebra.

Lemma 5 For structure constants γ_{ij}^{k} and δ_{ij}^{k} of LSD we have the following relationship:

$$\gamma_{ip}^{t}\gamma_{jk}^{p} = \gamma_{ip}^{t}\delta_{jk}^{p}$$

$$\delta_{ij}^{p}\delta_{pk}^{t} = \gamma_{ij}^{p}\delta_{pk}^{t}$$

$$\gamma_{ip}^{t}\gamma_{jk}^{p} - \gamma_{ij}^{p}\gamma_{pk}^{t} = \delta_{jp}^{t}\gamma_{ik}^{p} - \delta_{ji}^{p}\gamma_{pk}^{t}$$

$$\delta_{ip}^{t}\delta_{jk}^{p} - \delta_{ij}^{p}\delta_{pk}^{t} = \delta_{jp}^{t}\delta_{ik}^{p} - \delta_{ji}^{p}\delta_{pk}^{t}$$

Proposition 6 Let vector space S is Left-Symmetric Dialgebras with respect to $\dashv$ and $\vdash$. Then the following assertions are equivalent:

A) The Diassociative algebra $(S, \dashv, \vdash)$ is nilpotent,

B) The Left Symmetric Dialgebra $(S, \dashv, \vdash)$ is nilpotent.

Theorem 7 Any two-dimensional complex Left Symmetric Dialgebra S, can be included in one of the following isomorphism classes of algebras, $\forall a, b \in F$:

$LSD_2^1(a,b)$: $\quad e_1 \dashv e_2 = e_1, \quad e_2 \dashv e_2 = e_2, \quad e_2 \vdash e_1 = ae_1, \quad e_2 \vdash e_2 = be_1 + e_2, \quad a \neq 0;$

$LSD_2^2(b,c)$: $\quad e_1 \dashv e_2 = e_1, \quad e_2 \dashv e_2 = ce_1 + e_2, \quad e_2 \vdash e_2 = be_1 + e_2, \quad c \neq 0;$

$LSD_2^3(b)$: $\quad e_1 \dashv e_2 = e_1, \quad e_2 \dashv e_2 = e_2, \quad e_2 \vdash e_2 = be_1 + e_2;$

$LSD_2^4(c)$: $\quad e_2 \dashv e_2 = ce_1 + e_2, \quad e_2 \vdash e_1 = e_1, \quad e_2 \vdash e_2 = be_1 + e_2;$

$LSD_2^5(a,c)$: $\quad e_2 \dashv e_2 = ce_1 + e_2, \quad e_2 \vdash e_1 = ae_1, \quad e_2 \vdash e_2 = c(1-a)e_1 + e_2, \quad a \neq 1;$

$LSD_2^6(a)$: $\quad e_2 \dashv e_2 = e_2, \quad e_2 \vdash e_1 = ae_1, \quad e_2 \vdash e_2 = e_2, \quad a \neq 0;$

where LSD_2^i stands algebra number i in the list of two dimensional LSD.

Remark 8 It is not too difficult to see that for some values of parameters *a, b* and *c* from above listed Left Symmetric Dialgebras it can be obtained two dimensional Diassociative algebras described in [RRB].

Proposition 9 There are pair-wise isomorphic algebras among classes $LSD_2^i(a,b), 1 \leq i \leq 6,$ in theorem 2.7 for certain values of parameters. The table below describes the isomorphism conditions of algebras:

Classes	$LSD_2^6(a')$	$LSD_2^5(a',c')$	$LSD_2^4(c')$	$LSD_2^3(b')$	$LSD_2^2(b',c')$	$LSD_2^1(a',b')$
$LSD_2^1(a,b)$	not isom.	not isom.	not isom.	$a=0$	$a=1$, $c'=0$, $b'=\frac{c}{\alpha_1}$	$b'=\frac{\alpha_3(a-1)+b}{\alpha_1}$
$LSD_2^2(b,c)$	not isom.	not isom.	not isom.	$c=0$	$c'=\frac{c}{\alpha_1}$, $b'=\frac{b-\alpha_3}{\alpha_1}$	
$LSD_2^3(b)$	not isom.	not isom.	not isom.	$b'=\frac{b-\alpha_3}{\alpha_1}$		
$LSD_2^4(c)$	$c=0$, $a'=1$	$a'=1$	$c'=\frac{c-\alpha_3}{\alpha_1}$			
$LSD_2^5(a,c)$	$c=0$	$a'=\frac{a-\alpha_3}{\alpha_1}$, $c'=c$				
$LSD_2^6(a)$	$a'=a$					

while checking isomorphism of two classes we used basis change as follows:

$$e_1' = \alpha_1 e_1 + \alpha_2 e_2, \quad e_2' = \alpha_3 e_1 + \alpha_4 e_2$$

DERIVATIONS AND AUTOMORPHISMS OF LSD

Definition 10 Let S be a Left Symmetric Dialgebra. A derivation of S is a linear transformation $d: S \to S$ satisfying, $\forall x, y \in S$

$$d(x \dashv y) = d(x) \dashv y + x \dashv d(y), \qquad d(x \dashv y) = d(x) \dashv y + x \dashv d(y)$$

Let us consider Left Symmetric Dialgebra $(S, \dashv, \vdash)$ and linear transformations

$$ad_z(x) = \mathrm{x} \dashv \mathrm{z} - \mathrm{z} \vdash \mathrm{x}.$$

It can be easily verify that $ad_z(x)$ is a derivation of S. Those type derivations we call the Inner derivations of algebra S. The set of all inner derivations we denote by I(S). As known from classical algebras the subset of inner derivations I(S) is an ideal of Der(S).

Definition 11 A Left Symmetric Dialgebra S is called characteristically nilpotent if Der(S), the set of all derivations, is nilpotent as an algebra.

Lemma 10 The derivations and automorphisms of two dimensional complex Left Symmetric Dialgebras are as follows:

Classes	Derivations	Aut(S)
$LSD_2^1(a,b)$	$\begin{pmatrix} d_{11} & \frac{b}{a-1}d_{11} \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \frac{b(1-\alpha_1)}{a-1} \\ 0 & 1 \end{pmatrix}$
$LSD_2^2(b,c)$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & b(1-\alpha_1) \\ 0 & 1 \end{pmatrix}$
$LSD_2^3(b)$	$\begin{pmatrix} d_{11} & -bd_{11} \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & b(1-\alpha_1) \\ 0 & 1 \end{pmatrix}$
$LSD_2^4(c)$	$\begin{pmatrix} d_{11} & d_{21} \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & a(1-\alpha_1) \\ 0 & 1 \end{pmatrix}$
$LSD_2^5(a,c)$	$\begin{pmatrix} d_{11} & -ad_{11} \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & a(1-\alpha_1) \\ 0 & 1 \end{pmatrix}$
$LSD_2^6(a)$	$\begin{pmatrix} d_{11} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$

$d(e_1) = d_{11}e_1 + d_{12}e_2,\ \ d(e_2) = d_{21}e_1 + d_{22}e_2,$

Corollary 11 There is only one characteristic nilpotent Left Symmetric Dialgebra in dimension two.

Definition 12 Let S be a Left Symmetric Dialgebra (some subclasses of LSD are very important)

a) If $\forall x \in S,\ R_x\ and\ \ r_x$ are nilpotent then S is said to be **transitive** or complete.
b) If S has no ideals except itself and zero, then S is called **simple**.
c) If $\forall x, y \in A, R_x R_y = R_y R_x$, then S is called **Novikov algebra**.

Lemma 12 Let S be a two dimensional Left Symmetric Dialgebra described in Theorem 7, then

a) There is no commutative LSD algebra in two dimensional case;
b) All two dimensional LSD algebras are not transitive;
c) None of them is nilpotent;
d) There are two Novikov algebras amoung the two dimensional LSD described in theorem 7, they are: $LSD_2^2(b,c)$ and $LSD_2^3(b)$.

ROTA-BAXTER OPERATORS OF LSD

Rota-Baxter operator was introduced by G. Baxter in 1960 to solve an analytic problems. Later it was intensively studied in many fields of mathematics, mathematical physics and quantum field theory. Up to now, most of studies on Rota-Baxter operators was on the Associative, Lie, Left Symmetric Algebras with different weights.

Definition 13. Let S be an (associative or non-associative) algebra over a field F. S linear operator $R: S \to S$ is called a Rota-Baxter operator of weight $\alpha \epsilon F$ on S if R satisfies the following relation:

$$R(x)R(y) + \alpha R(xy) = R\big(R(x)y + xR(y)\big), \qquad \forall x, y \in S$$

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of LSD $(S, \dashv, \vdash)$ and

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k, \qquad e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^{k} e_k$$

Then any Rota-Baxter operator R on S can be presented by a matrix (r_{ij}), where $R(e_i) = \sum_{j=1}^{n} r_{ij}\, e_j$.

Proposition 13. The Rota-Baxter operators on 2 dimensional Left Symmetric Dialgebras are given in the following table (any parameter belongs to complex field)

2 dim LSD	Rota-Baxter Operators
$LSD_2^1(a,b)$	$\begin{pmatrix} 0 & r_{12} \\ r_{21} & 0 \end{pmatrix}, \begin{pmatrix} r_{12} & -\sqrt{8} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
$LSD_2^2(b,c)$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
$LSD_2^3(b)$	$\begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
$LSD_2^4(c)$	$\begin{pmatrix} r_{11} & 0 \\ r_{21} & 0 \end{pmatrix}$
$LSD_2^5(a,c)$	$\begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ r_{21} & 0 \end{pmatrix}$
$LSD_2^6(a)$	$\begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix}$

"CHECK LSD" MAPLE PROGRAM

```
checkLSD := proc(A, B, n)
    local i, j, k, p, t;
    for i to n do
        for j to n do
            for k to n do
                for p to n do
                    if simplify(sum(A[i, p, t]*A[j, k, p]− A[i, p, t]
                     *B[j, k, p], t = 1 ..n)) <> 0 then
                        RETURN("Input is NOT a LSD ---the LSD
                        (", i, j, p, ") is not zero")
                     end if;
                    if simplify(sum(B[i, j, p]*B[p, k, t]− A[i, j, p]
                     *B[p, k, t], t = 1 ..n)) <> 0 then
                        RETURN("Input is NOT a LSD ---the LSD
                        (", i, j, p, ") is not zero")
                     end if;
                    if simplify(sum(A[i, p, t]*A[j, k, p]− A[i, j, p]
                     *A[p, k, t] − B[j, p, t]*A[i, k, p] + B[j, i, p]
                     *A[p, k, t], t = 1 ..n)) <> 0 then
                        RETURN("Input is NOT a LSD ---the LSD
                        (", i, j, p, ") is not zero")
                     end if;
                    if simplify(sum(B[i, p, t]*B[j, k, p]− B[i, j, p]
                     *B[p, k, t] − B[j, p, t]*B[i, k, p] + B[j, i, p]
                     *B[p, k, t], t = 1 ..n)) <> 0 then
                        RETURN("Input is NOT a LSD ---the LSD
                        (", i, j, p, ") is not zero")
                     end if;
                    end do
                end do
            end do
        end do;
    print("Yes, input IS a Left Symmetric Dialgebra")
end proc
```

Example. As we mentioned above this Maple programme concerns to check whether given/ obtained algebra is LSD or not. Let us consider one of the algebras from Theorem and check:

```
LSD1.1 :=array (sparse, 1..2, 1..2, 1..2, [(1,2,1)=1, (2,2,2)=1]);

LSD1.2 :=array (sparse, 1..2, 1..2, 1..2, [(2,1,1)=a, (2,2,1)=b, (2,2,2)=1]);

checkLSD(LSD1.1, LSD1.2, 2);

"Yes, input is a Left-Symmetric Dialgebra".
```

Since Maple program confirmed that obtained algebra is satisfied all identities of LSD so we may include it into list of algebras in corresponding dimension.

Future Research Opportunities on structure theory of LSD

1. Classification of low dimensional LSD over complex numbers.
2. Investigate properties of nilpotent LSD.
3. Explore Simple and semi-simple LSD.
4. Examine Radicals of Left Symmetric Dialgebras.
5. Study Rota-Baxter Operators on Left Symmetric Dialgebras.
6. Search for more examples and applications of LSD to other branches of math and sciences.
7. Discover Derivations (Inner derivations) and Automorphisms of LSD.
8. Introduce and investigate Graded Left Symmetric Dialgebras.
9. Establish software programmes in order to simplify calculation works.
10. Many more …

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thank you
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