

2-Local derivations on von Neumann algebras and AW^* - algebras

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Outline

1 Von Neumann algebras

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- 2 Derivation
 - Derivation and local derivation
 - A Kowalski-Słodkowski theorem for 2-local $*$ -homomorphisms on von Neumann algebras
 - 2-local derivations on $B(H)$

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 - Sketch of the proof

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Von Neumann algebras

Let H be a Hilbert space over the field \mathbb{C} of complex numbers, and let $B(H)$ be the algebra of all bounded linear operators on H . Denote by $\mathbf{1}$ the identity operator on H , and let $P(H) = \{p \in B(H) : p = p^2 = p^*\}$ be the lattice of projections in $B(H)$. Consider a von Neumann algebra M on H , i.e. a $*$ -subalgebra of $B(H)$ closed in the weak operator topology and containing the operator $\mathbf{1}$. Denote by $\|\cdot\|_M$ the operator norm on M . The set $P(M) = P(H) \cap M$ is a complete orthomodular lattice with respect to the natural partial order on $M_h = \{x \in M : x = x^*\}$, generated by the cone M_+ of positive operators from M .

Von Neumann algebras

Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I , II_1 (finite), II_∞ (properly infinite, semifinite) and III (purely infinite).

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Derivation

Given an algebra \mathcal{A} , a linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a **derivation**, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule).

Derivation

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Each element $a \in \mathcal{A}$ implements a derivation D_a on \mathcal{A} defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be **inner derivations**.

Local derivation

There exist various types of linear operators which are close to derivations. Recall that a linear map Δ of \mathcal{A} is called a **local derivation** if for each $x \in \mathcal{A}$, there exists a derivation $D : \mathcal{A} \rightarrow \mathcal{A}$, depending on x , such that $\Delta(x) = D(x)$. This notion was introduced in 1990 independently by Kadison ^a and Larson and Sourour ^b.

^aR. V. Kadison, Local derivations, J. Algebra. 130 (1990), 494–509.

^bD. R. Larson, A. R. Sourour, Local derivations and local automorphisms of $B(X)$, Proc. Sympos. Pure Math. 51. Providence, Rhode Island, 1990, Part 2, 187–194.

Local derivation

Kadison had proved that every norm continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. The same result was obtained for the algebra of all bounded linear operators acting on a Banach space by Larson and Sourour.

2-local derivation

Recall that a map

$$\Delta : \mathcal{A} \rightarrow \mathcal{A}$$

(not linear in general) is called a **2-local derivation** if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

2-local derivation

The notions of **local** and **2-local automorphisms** of algebras are defined in the same way.

2-local derivation

The notions of **local and 2-local automorphisms** of algebras are defined in the same way.

Any derivation (resp. automorphism) is a local and a 2-local derivation (resp. automorphism), but the converse is not true in general.

Example

Consider an algebra upper-triangular 2×2 -matrix

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Define operator Δ on \mathcal{A} by

$$\Delta(A) = \begin{cases} 0, & \text{if } a_{11} \neq a_{22}, \\ \begin{pmatrix} 0 & 2a_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } a_{11} = a_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation^a.

^aJ. H. Zhang, H. X. Li, 2-Local derivations on digraph algebras, Acta Math. Sinica, Chinese series 49 (2006), 1401–1406.

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A Kowalski-Słodkowski theorem for 2-local $*$ -homomorphisms on von Neumann algebras

The Gleason-Kahane-Żelazko theorem

[A.M. Gleason, A characterization of maximal ideals, *J. Analyse Math.* 19, 171-172 (1967)], [J.P. Kahane, W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, *Studia Math.* 29, 339-343 (1968)],

a fundamental contribution in the theory of Banach algebras, asserts that every unital linear functional F on a complex unital Banach algebra A such that, $F(a)$ belongs to the spectrum, $\sigma(a)$, of a for every $a \in A$, is multiplicative. In modern terminology, this is equivalent to say that every unital linear local homomorphism from a unital complex Banach algebra A into \mathbb{C} is multiplicative.

A Kowalski-Słodkowski theorem for 2-local $*$ -homomorphisms on von Neumann algebras

After the Gleason-Kahane-Żelazko theorem was established, Kowalski and Słodkowski

[S.Kowalski, Z. Słodkowski, A characterization of multiplicative linear functionals in Banach algebras, *Studia Math.* 67, 215-223 (1980)]

showed that at the cost of requiring the local behavior at two points, the condition of linearity can be dropped, that is, suppose A is a complex Banach algebra (not necessarily commutative nor unital), then every (not necessarily linear) mapping $T : A \rightarrow \mathbb{C}$ satisfying $T(0) = 0$ and $T(x - y) \in \sigma(x - y)$, for every $x, y \in A$, is multiplicative and linear.

A Kowalski-Słodkowski theorem for 2-local $*$ -homomorphisms on von Neumann algebras

According to the above notation, the result established by Kowalski and Słodkowski proves that every (not necessarily linear) 2-local homomorphism T from a (not necessarily commutative nor unital) complex Banach algebra A into the complex field \mathbb{C} is linear and multiplicative. Consequently, every (not necessarily linear) 2-local homomorphism T from A into a commutative C^* -algebra is linear and multiplicative.

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2-local derivation on $B(H)$. Separabel case

The notion of 2-local derivations it was introduced in 1997 by P. Šemrl ^a and in this paper he described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H . A similar description for the finite-dimensional case appeared later by S. O. Kim and J. S. Kim ^b

^aŠemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125 (1997) 2677–2680.

^bS. O. Kim, J. S. Kim, Local automorphisms and derivations on M_n , Proc. Amer. Math. Soc. 132 (2004) 1389–1392.

2-local derivation on $B(H)$. Separabel case

The methods of the proofs above mentioned results essentially based the fact that an algebra $B(H)$ is single generated.
For example O. Kim and J. S. Kim used the following two matrices

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$$A = \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2^n} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

2-local derivation on $B(H)$. General case

In 2012 ^a we suggested a new technique and have generalized the above mentioned results for arbitrary Hilbert spaces. Namely we considered 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

^aSh. A. Ayupov, K. K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl. 395 (2012) 15–18.

2-local derivation on $B(H)$. General case

Our proof essentially use existence a faithful normal semi-finite trace on $B(H)$. Namely, the main ingredient of our paper is the following identity

$$\operatorname{tr}(\Delta(x)y) = -\operatorname{tr}(x\Delta(y)) \quad (1)$$

for all $x \in B(H)$, and for finite-dimensional operator $y \in B(H)$, where tr is the canonical trace on $B(H)$.

Semi-finite von Neumann algebra

A similar result for 2-local derivations on finite von Neumann algebras was obtained by Sh. A. Ayupov and etal. ^a. Finally, Sh. A. Ayupov and F. N. Arzikulov^b extended all above results and give a short proof of this result for arbitrary semi-finite von Neumann algebras.

^aSh. A. Ayupov, K. K. Kudaybergenov, B. O. Nurjanov, A. K. Alauatdinov, Local and 2-local derivations on noncommutative Arens algebras, *Mathematica Slovaca*. 64 (2014) 1–10.

^bSh. A. Ayupov, F. N. Arzikulov, 2-local derivations on semi-finite von Neumann algebras, *Glasgow Math. Jour.* 56 (2014) 9–12.

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Main result

The following theorem is the main result of this section.

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Theorem 3.1

Let M be an arbitrary von Neumann algebra. Then any 2-local derivation $\Delta : M \rightarrow M$ is a derivation.

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Sketch of the proof

For a self-adjoint subset $S \subseteq M$ denote by S' is the commutant of S , i.e.

$$S' = \{y \in B(H) : xy = yx, \forall x \in S\}.$$

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$$S' = \{y \in B(H) : xy = yx, \forall x \in S\}.$$

Let $g \in M$ be a self-adjoint element and let $\mathcal{W}^*(g) = \{g\}''$ be the abelian von Neumann algebra generated by the element g . Then there exists an element $a \in M$ such that

$$\Delta(x) = ax - xa$$

for all $x \in \mathcal{W}^*(g)$. In particular, Δ is additive on $\mathcal{W}^*(g)$.

Sketch of the proof

Let $P(M)$ denote the lattice of all projections of the von Neumann algebra M . Recall that a map $m : P(M) \rightarrow \mathbb{C}$ is called a signed measure (or charge) if $m(e_1 + e_2) = m(e_1) + m(e_2)$ for arbitrary orthogonal projections e_1, e_2 in M . A signed measure m is said to be bounded if $\sup\{|m(e)| : e \in P(M)\}$ is finite.

Sketch of the proof

We need the following version of the Gleason's Theorem^a.

^aL.J. Bunce, J.D. Maitland Wright, The Mackey-Gleason problem, Bull. Amer. Math. Soc., 26 (1992) 288–293.

Sketch of the proof

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Gleason Theorem

Let \mathcal{A} be a von Neumann algebra with no direct summand of Type I_2 . Then each complex-valued finitely additive measure on $P(\mathcal{A})$ extends to a bounded linear functional on A .

Sketch of the proof

The following lemma is one of the key steps in the proof of the main result.

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Lemma 3.2

Let M be an infinite von Neumann algebra. The restriction $\Delta|_{M_{sa}}$ of the 2-local derivation Δ onto the set M_{sa} of all self-adjoint of M is additive.

Sketch of the proof

Lemma 3.3

There exists an element $a \in M$ such that $\Delta(x) = D_a(x) = ax - xa$ for all $x \in M_{sa}$.

Sketch of the proof

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In order to prove this Lemma we consider the extension $\tilde{\Delta}$ of $\Delta|_{M_{sa}}$ on M defined by:

$$\tilde{\Delta}(x_1 + ix_2) = \Delta(x_1) + i\Delta(x_2), \quad x_1, x_2 \in M_{sa}.$$

Sketch of the proof

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$$\tilde{\Delta}(x_1 + ix_2) = \Delta(x_1) + i\Delta(x_2), \quad x_1, x_2 \in M_{sa}.$$

Lemma 3.4

If $\Delta|_{M_{sa}} \equiv 0$, then $\Delta \equiv 0$.

For details of the proof we refer to
Sh.A.Ayupov, K. K.Kudaybergenov, "2-Local derivations on von
Neumann algebras" POSITIVITY, 19, 2015, No.3, 445-455.

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AW^* -algebras

The notion of AW^* -algebras was introduced by Kaplansky as an abstract generalization of von Neumann algebras. Namely, AW^* -algebra is a C^* -algebra such that the left annihilator of any subset is a principal left ideal generated by a projection. He showed that much of the "non spatial theory" of von Neumann algebras can be extended to AW^* -algebras.

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Every von Neumann algebra is a AW^* -algebra but the converse is not true as was shown by Dixmier with an abelian example [J. Dixmier, Sur certains espaces consideres par M.H.Stone, Summa Brasil Math. 2 (1951), 151-182].

AW^* -algebras

Kaplansky proved that an AW^* -algebra of type I is a von Neumann algebra if and only if its center is a von Neumann algebra and conjectured that this is true for general AW^* -algebras. But in 1970 Takenouchi and Dyer independently showed this to be false by providing examples of type III AW^* -algebras which are not von Neumann algebras
[O. Takenouchi, A non- W^* , AW^* -factor, Lect. Notes Math., vol. 650 (1978), 135-139],
[J.Dyer, Concerning AW^* -algebras, Notices Amer. Math. Soc., 17 (1970), 788].

AW^* -algebras

In the present section we extend our above results concerning 2-local derivations to the case of arbitrary AW^* -algebras. First we consider 2-local derivations on matrix algebras over unital semi-prime Banach algebras. Namely, we prove that if \mathcal{A} is a unital semi-prime Banach algebra with the inner derivation property then any 2-local derivation on the algebra $M_{2^n}(\mathcal{A})$, $n \geq 2$, is a derivation. We apply this result to AW^* -algebras and prove that any 2-local derivation on an arbitrary AW^* -algebra is a derivation.

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2-Local derivations on matrix algebras

If $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ is a 2-local derivation, then from the definition it easily follows that Δ is homogenous. At the same time,

$$\Delta(x^2) = \Delta(x)x + x\Delta(x) \quad (2)$$

for each $x \in \mathcal{A}$.

2-Local derivations on matrix algebras

In the paper of M. Brešar

[M. Brešar, Jordan derivations on semi-prime rings. Proc. Amer. Math. Soc., 104 (1988), 1003-1006]

it is proved that any **Jordan derivation** (i.e. a linear map satisfying the above equation) on a semi-prime algebra is a derivation. Therefore, in the case semi-prime algebras in order to prove that a 2-local derivation $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation, it is sufficient to prove that $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ is additive.

2-Local derivations on matrix algebras

We say that an algebra \mathcal{A} has the **inner derivation property** if every derivation on \mathcal{A} is inner. Recall that an algebra \mathcal{A} is said to be **semi-prime** if $a\mathcal{A}a = 0$ implies that $a = 0$.

Let $M_n(\mathcal{A})$ be the algebra of $n \times n$ -matrices over \mathcal{A} and assume that $n \geq 2$.

2-Local derivations on matrix algebras

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Let $M_n(\mathcal{A})$ be the algebra of $n \times n$ -matrices over \mathcal{A} and assume that $n \geq 2$.

Lemma 4.1

Let \mathcal{A} be a unital Banach algebra with the inner derivation property. Then the algebra $M_n(\mathcal{A})$ also has the inner derivation property.

2-Local derivations on matrix algebras

The following theorem is the main result of this section.

2-Local derivations on matrix algebras

The following theorem is the main result of this section.

Theorem 4.2

Let \mathcal{A} be a unital semi-prime Banach algebra with the inner derivation property and let $M_{2^n}(\mathcal{A})$ be the algebra of $2^n \times 2^n$ -matrices over \mathcal{A} . Then any 2-local derivation Δ on $M_{2^n}(\mathcal{A})$ is a derivation.

2-Local derivations on matrix algebras

The proof of Theorem 4.2. is rather technical and consists of two steps. For details we refer to our paper [Sh.A.Ayupov, K.K.Kudaybergenov, 2-Local derivations on matrix algebras over semi-prime rings and on AW^* -algebras. arXiv:1411.2471v1 [Math.OA]]

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

In the second step of our proof we show that if a 2-local derivation Δ on a matrix algebra equals to zero on all diagonal matrices and on the linear span of matrix units, then it is identically zero on the whole algebra.

2-Local derivations on matrix algebras

The condition on the algebra \mathcal{A} to be a Banach algebra was applied in the proof only for the invertibility of elements of the forms $\mathbf{1} + x$, where $x \in \mathcal{A}$, $\|x\| < 1$. In this connection the following question naturally arises.

2-Local derivations on matrix algebras

The condition on the algebra \mathcal{A} to be a Banach algebra was applied in the proof only for the invertibility of elements of the forms $\mathbf{1} + x$, where $x \in \mathcal{A}$, $\|x\| < 1$. In this connection the following question naturally arises.

Problem 4.3.

Does Theorem 4.2. hold for arbitrary (not necessarily normed) algebra \mathcal{A} with the inner derivation property?

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An application to AW^* -algebras

In this section we apply Theorem 4.2. to the description of 2-local derivations on AW^* -algebras.

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Theorem 5.1

Let \mathcal{A} be an arbitrary AW^* -algebra. Then any 2-local derivation Δ on \mathcal{A} is a derivation.

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Proof

Proof. Let us first note that any AW^* -algebra is semi-prime. It is also known that AW^* -algebra has the inner derivation property [D. Olesen, Derivations AW^* -algebras are inner, Pacific J. Math., 53, 555-561 (1974)].

Proof

Let z be a central projection in \mathcal{A} . Since $D(z) = 0$ for an arbitrary derivation D , it is clear that $\Delta(z) = 0$ for any 2-local derivation Δ on \mathcal{A} . Take $x \in \mathcal{A}$ and let D be a derivation on \mathcal{A} such that $\Delta(zx) = D(zx)$, $\Delta(x) = D(x)$. Then we have $\Delta(zx) = D(zx) = D(z)x + zD(x) = z\Delta(x)$. This means that every 2-local derivation Δ maps $z\mathcal{A}$ into $z\mathcal{A}$ for each central projection $z \in \mathcal{A}$. So, we may consider the restriction of Δ onto $e\mathcal{A}$. Since an arbitrary AW^* -algebra can be decomposed along a central projection into the direct sum of an abelian AW^* -algebra, and AW^* -algebras of type I_n , $n \geq 2$, type I_∞ , type II and type III, we may consider these cases separately.

Proof

Let \mathcal{A} be an abelian AW^* -algebra. It is well-known that any derivation on a such algebra is identically zero. Therefore any 2-local derivation on an abelian AW^* -algebra is also identically zero.

Proof

Let \mathcal{A} be an abelian AW^* -algebra. It is well-known that any derivation on a such algebra is identically zero. Therefore any 2-local derivation on an abelian AW^* -algebra is also identically zero.

If \mathcal{A} is an AW^* -algebra of type I_n , $n \geq 2$, with the center $Z(\mathcal{A})$, then it is isomorphic to the algebra $M_n(Z(\mathcal{A}))$. By Theorem 4.2. there exists a derivation D on $\mathcal{A} \cong M_n(Z(\mathcal{A}))$ such that $\Delta \equiv D$. So, Δ is a derivation.

Proof

Let the AW^* -algebra \mathcal{A} have one of the types I_∞ , II or III. Then the halving Lemmata for type I_∞ , type II and type III

AW^* -algebras from

[I. Kaplansky, Projections in Banach algebras, Ann. Math. 53, 235-249 (1951)],

imply that the unit of the algebra \mathcal{A} can be represented as a sum of mutually equivalent orthogonal projections e_1, e_2, e_3, e_4

from \mathcal{A} . Then the map $x \mapsto \sum_{i,j=1}^4 e_i x e_j$ defines an isomorphism

between the algebra \mathcal{A} and the matrix algebra $M_4(\mathcal{B})$, where $\mathcal{B} = e_{1,1} \mathcal{A} e_{1,1}$. Therefore Theorem 4.2. implies that any 2-local derivation on \mathcal{A} is a derivation. The proof is complete.

THANKS FOR YOUR ATTENTION!