# Local and 2-local derivations and automorphisms of finite-dimensional Lie algebras

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# Introduction

Given an algebra $\mathcal{A}$, a linear mapping $T : \mathcal{A} \to \mathcal{A}$ is called an automorphism (respectively a derivation) if $T(ab) = T(a)T(b)$ (respectively, $T(ab) = T(a)b + aT(b)$) for all $a, b \in \mathcal{A}$.

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A linear operator $\Delta : \mathcal{A} \to \mathcal{A}$ is called a local automorphism (respectively, a local derivation) if for every $x$ in $\mathcal{A}$ there exists an automorphism $\alpha_x$ (respectively, a derivation $d_x$) on $\mathcal{A}$ depending on $x$, such that $\Delta(x) = \alpha_x(x)$ (respectively, $\Delta(x) = d_x(x)$).

# Introduction

A mapping $\Delta : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-local automorphism (respectively, a 2-local derivation) on $\mathcal{A}$, if for every $x, y \in \mathcal{A}$, there exists an automorphism $\alpha_{x,y} : \mathcal{A} \to \mathcal{A}$ (respectively, a derivation $d_{x,y}$ on $\mathcal{A}$) such that $\Delta(x) = \alpha_{x,y}(x)$ and $\Delta(y) = \alpha_{x,y}(y)$ (respectively, $\Delta(x) = d_{x,y}(x)$ and $\Delta(y) = d_{x,y}(y)$).

# Introduction

The main problems concerning the above notions are to find conditions under which every local (or 2-local) automorphism or derivation automatically becomes an automorphism (respectively, a derivation), and also to present examples of algebras with local and 2-local automorphisms (respectively, derivations) that are not automorphisms (respectively, derivations).

# Introduction

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In this talk we start with 2-local derivations and automorphisms and then pass to local derivations..

# Introduction

The notion of 2-local derivations has been introduced by P. Šemrl in
[P. Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc., 125 (1997) 2677–2680]
and there the author described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an infinite-dimensional separable Hilbert space $H$. Namely he has proved that every 2-local derivation on $B(H)$ is a derivation. A similar description for the finite-dimensional case appeared later in
[S. O. Kim, J. S. Kim, Local automorphisms and derivations on $M_n$, Proc. Amer. Math. Soc., 132 (2004) 1389–1392].

# Introduction

In [Sh. A. Ayupov, K. K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl., 395 (2012) 15–18] a new technique has been proposed which enabled the authors to generalize the above mentioned results for arbitrary Hilbert spaces $H$. Namely it was proved that on the algebra $B(H)$ for an arbitrary Hilbert space $H$ (no separability is assumed) every 2-local derivation is a derivation. A similar result for 2-local derivations on finite von Neumann algebras was obtained in [Sh. A. Ayupov, K. K. Kudaybergenov, B. O. Nurjanov, A. K. Alauadinov, Local and 2-local derivations on noncommutative Arens algebras, Math. Slovaca, 64 (2014) 423–432].

# Introduction

In
[ Sh. A. Ayupov, F. N. Arzikulov, 2-local derivations on semi-finite von Neumann algebras, Glasgow Math. Jour. 56 (2014) 9–12]
the authors have extended all above results for arbitrary semi-finite von Neumann algebras. Finally, in
[Sh. A. Ayupov, K.K. Kudaybergenov, 2-local derivations on von Neumann algebras, Positivity, 19 (2015) no 3, 445–455]
it was proved that for any purely infinite von Neumann algebra $M$ every 2-local derivation on $M$ is a derivation. This completed the solution of the above problem for arbitrary von Neumann algebras.

# Introduction

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The motivation to study the case came out from discussions made with the Fields Medalist Professor E. Zelmanov (University of California, San Diego) during USA–Uzbekistan Conference held at the California State University, Fullerton, on May, 2014. The author gratefully acknowledges him for discussions.
Let us start with some motivations for consideration of local and 2-local automorphisms of Banach algebras.

# A Kowalski-Słodkowski theorem for 2-local *-homomorphisms on von Neumann algebras

The Gleason-Kahane-Żelazko theorem [A.M. Gleason, A characterization of maximal ideals, J. Analyse Math. 19, 171-172 (1967)], [J.P. Kahane, W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, Studia Math. 29, 339-343 (1968)], a fundamental contribution in the theory of Banach algebras, asserts that every unital linear functional $F$ on a complex unital Banach algebra $A$ such that, $F(a)$ belongs to the spectrum, $\sigma(a)$, of $a$ for every $a \in A$, is multiplicative. In modern terminology, this is equivalent to say that every unital linear local homomorphism from a unital complex Banach algebra $A$ into $\mathbb{C}$ is multiplicative.

# A Kowalski-Słodkowski theorem for 2-local *-homomorphisms on von Neumann algebras

After the Gleason-Kahane-Żelazko theorem was established, Kowalski and Słodkowski
[S.Kowalski, Z. Słodkowski, A characterization of multiplicative linear functionals in Banach algebras, Studia Math. 67, 215-223 (1980)]
showed that at the cost of requiring the local behavior at two points, the condition of linearity can be dropped, that is, suppose $A$ is a complex Banach algebra (not necessarily commutative nor unital), then every (not necessarily linear) mapping $T : A \to \mathbb{C}$ satisfying $T(0) = 0$ and $T(x - y) \in \sigma(x - y)$, for every $x, y \in A$, is multiplicative and linear.

# A Kowalski-Słodkowski theorem for 2-local *-homomorphisms on von Neumann algebras

According to the above notation, the result established by Kowalski and Słodkowski proves that every (not necessarily linear) 2-local homomorphism $T$ from a (not necessarily commutative nor unital) complex Banach algebra $A$ into the complex field $\mathbb{C}$ is linear and multiplicative. Consequently, every (not necessarily linear) 2-local homomorphism $T$ from $A$ into a commutative C*-algebra is linear and multiplicative.

# Outline

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Recall, that Lie algebra is vector space $\mathcal{L}$ over some field $\mathbb{F}$ together with a binary operation $[.,.]$ called the Lie brackets, which satisfy the following axioms:

(i) Bilinearity:
$[ax+by,z]=a[x,z]+b[y,z], [z,ax+by]=a[z,x]+b[z,y]$ for all scalars $a,b$ in $\mathbb{F}$ and all elements $x,y,z$ in $\mathcal{L}$.

(ii) Alternating on $\mathcal{L}$: $[x,x]=0$ for all $x$ in $\mathcal{L}$.

(iii) The Jacobi identity: $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$ for all $x,y,z$ in $\mathcal{L}$.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

It should be noted that the bilinearity and alternating properties imply anticommutativity, i.e. $[x, y] = -[y, x]$ for all $x, y$, in $\mathcal{L}$, while anticommutativity implies the alternating property only if the field's characteristic is not 2.
Let $\mathcal{L}$ be a Lie algebra. The center of $\mathcal{L}$ is denoted by $Z(\mathcal{L})$ :

$$Z(\mathcal{L}) = \{x \in \mathcal{L} : [x, y] = 0, \forall\, y \in \mathcal{L}\}.$$

A Lie algebra $\mathcal{L}$ is called nilpotent if $\mathcal{L}^k = \{0\}$ for some positive integer $k$, where $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^k = [\mathcal{L}^{k-1}, \mathcal{L}]$, $k \geq 1$.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

A Lie algebra $\mathcal{L}$ is said to be solvable if $\mathcal{L}^{(k)} = \{0\}$ for some integer $k$, where $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(k)} = [\mathcal{L}^{(k-1)}, \mathcal{L}^{(k-1)}]$, $k \geq 1$. Any Lie algebra $\mathcal{L}$ contains a unique maximal solvable ideal, called the radical of $\mathcal{L}$ and denoted $\mathrm{Rad}\mathcal{L}$. A non trivial Lie algebra $\mathcal{L}$ is called semi-simple if $\mathrm{Rad}\mathcal{L} = 0$. That is equivalent to requiring that $\mathcal{L}$ have no nonzero abelian ideals.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Given a vector space $V$, let $\mathfrak{gl}(V)$ denote the Lie algebra of all linear endomorphisms of $V$. A representation of a Lie algebra $\mathcal{L}$ on $V$ is a Lie algebra homomorphism $\rho : \mathcal{L} \to \mathfrak{gl}(V)$. For example, $\mathrm{ad} : \mathcal{L} \to \mathfrak{gl}(\mathcal{L})$ given by $\mathrm{ad}(x)(y) = [x, y]$ is a representation of $\mathcal{L}$ on the vector space $\mathcal{L}$ called the adjoint representation. If $V$ is a finite-dimensional vector space then the representation $\rho$ is said to be finite-dimensional.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Let $\mathcal{L}$ be a Lie algebra and let $\rho : \mathcal{L} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathcal{L}$. Then the map $\tau : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ defined by

$$\tau(x, y) = \mathrm{tr}(\rho(x)\rho(y))$$

is a symmetric bilinear form on $\mathcal{L}$ called the trace form on $\mathcal{L}$ relative to $\rho$, where tr denotes the trace of a linear operator. In particular, for $V = \mathcal{L}$ and $\rho = \mathrm{ad}$ the corresponding trace form is called the Killing form and it is denoted by $\langle \cdot, \cdot \rangle$.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

The Killing form has the following ad$\mathcal{L}$-invariancy property which we make use in the study of 2-local derivations on semi-simple Lie algebras:

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle \text{ for all } x, y, z \in \mathcal{L}.$$

Another importance of the Killing form is the following property. A Lie algebra $\mathcal{L}$ is semi-simple if and only if its Killing form is non degenerate, i.e. $\langle x, y\rangle = 0$ for all $y \in \mathcal{L}$ implies that $x = 0$.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Recall that a derivation on a Lie algebra $\mathcal{L}$ is a linear map $\delta : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz law, that is,

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathcal{L}$. The set of all derivations of a Lie algebra $\mathcal{L}$ is a Lie algebra with respect to the commutation operation and it is denoted by $\mathrm{Der}\mathcal{L}$. For any $a \in \mathcal{L}$, $\mathrm{ad}(a)$ is a derivation and derivations of this form are called inner derivation. The set of all inner derivations of $\mathcal{L}$ denoted $\mathrm{ad}L$ is an ideal in $\mathrm{Der}\mathcal{L}$. It is well known that any derivation on a finite-dimensional semi-simple Lie algebra is inner.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Recall once more that a map $T:\mathcal{L}\to\mathcal{L}$ (not linear in general) is called a 2-local derivation if for every $x, y\in\mathcal{L}$, there exists a derivation $\delta_{x,y}:\mathcal{L}\to\mathcal{L}$ (depending on $x, y$) such that $T(x)=\delta_{x,y}(x)$ and $T(y)=\delta_{x,y}(y)$.

Since any derivation on a finite-dimensional semi-simple Lie algebra $\mathcal{L}$ is inner, it follows that for such algebras the above definition of 2-local derivation is reformulated as follows. A map $T:\mathcal{L}\to\mathcal{L}$ is called a 2-local derivation on $\mathcal{L}$, if for any two elements $x$, $y\in\mathcal{L}$ there exists an element $a_{x,y}\in\mathcal{L}$ (depending on $x, y$) such that

$$T(x)=[a_{x,y},x],\ T(y)=[a_{x,y},y].$$

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Henceforth, given a 2-local derivation $T$, the symbol $a_{x,y}$ will denote an element from $\mathcal{L}$ satisfying $T(x) = [a_{x,y}, x]$ and $T(y) = [a_{x,y}, y]$.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

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The main result of this section is given as follows.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

Henceforth, given a 2-local derivation $T$, the symbol $a_{x,y}$ will denote an element from $\mathcal{L}$ satisfying $T(x) = [a_{x,y}, x]$ and $T(y) = [a_{x,y}, y]$.

The main result of this section is given as follows.

**Theorem 2.1.**

Let $\mathcal{L}$ be an arbitrary finite-dimensional semi-simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. Then any (not necessarily linear) 2-local derivation $T : \mathcal{L} \to \mathcal{L}$ is a derivation.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

See for details,
[Sh. A. Ayupov, K. K. Kudaybergenov, I. S. Rakhimov, 2-Local derivations on finite-dimensional Lie algebras, Linear Algebra and its Applications, 474 (2015), 1-11.]

# 2-local derivations on finite-dimensional semi-simple Lie algebras

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# 2-local derivations on finite-dimensional semi-simple Lie algebras

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One of key auxiliary result for the proof of the above theorem is the following

**Proposition 2.2.**

Let $T$ be a 2-local derivation on a finite-dimensional semi-simple Lie algebra $\mathcal{L}$. Then $T$ is linear.

# 2-local derivations on finite-dimensional semi-simple Lie algebras

**Proof**

Let $x, y, z \in \mathcal{L}$ be arbitrary elements. Taking into account $\mathrm{ad}\mathcal{L}$-invariancy of the Killing form we obtain that

$$\begin{aligned}
\langle T(x+y), z\rangle &= \langle [a_{x+y,z}, x+y], z\rangle = -\langle [x+y, a_{x+y,z}], z\rangle = \\
&= -\langle x+y, [a_{x+y,z}, z]\rangle = -\langle x, [a_{x+y,z}, z]\rangle - \\
&- \langle y, [a_{x+y,z}, z]\rangle = -\langle x, T(z)\rangle - \langle y, T(z)\rangle = \\
&= -\langle x, [a_{x,z}, z]\rangle - \langle y, [a_{y,z}, z]\rangle = \\
&= -\langle [x, a_{x,z}], z\rangle - \langle [y, a_{y,z}], z\rangle = \\
&= \langle [a_{x,z}, x], z\rangle + \langle [a_{y,z}, y], z\rangle = \\
&= \langle T(x), z\rangle + \langle T(y), z\rangle = \langle T(x)+T(y), z\rangle,
\end{aligned}$$

# 2-local derivations on finite-dimensional semi-simple Lie algebras

**Proof**

i.e.

$$\langle T(x+y), z\rangle = \langle T(x)+T(y), z\rangle.$$

Since the Killing form $\langle \cdot, \cdot \rangle$ is non-degenerate, the last equality implies that $T(x+y) = T(x)+T(y)$ for all $x, y \in \mathcal{L}$.

Finally,

$$T(\lambda x) = [a_{\lambda x, x}, \lambda x] = \lambda [a_{\lambda x, x}, x] = \lambda T(x).$$

So, $T$ is linear. The proof is complete.

# Outline

# 2-local derivations on nilpotent Lie algebras

In this section we give examples of 2-local derivations on nilpotent Lie algebras which are not derivations.

Note that a linear operator $\delta$ on a Lie algebra $\mathcal{L}$ such that $\delta|_{[\mathcal{L},\mathcal{L}]} \equiv 0$ and $\delta(\mathcal{L}) \subseteq Z(\mathcal{L})$ is a derivation. Indeed, for every $x, y \in \mathcal{L}$ we have

$$\delta([x,y]) = 0 = [\delta(x), y] + [x, \delta(y)].$$

# 2-local derivations on nilpotent Lie algebras

**Theorem 2.3.**

Let $\mathcal{L}$ be a $n$-dimensional Lie algebra with $n \geq 2$. Suppose that

(i) $\dim[\mathcal{L}, \mathcal{L}] \leq n - 2$;

(ii) the center $Z(\mathcal{L})$ of $\mathcal{L}$ is non trivial.

Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

# 2-local derivations on nilpotent Lie algebras

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Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

**Proof**

Let us consider a decomposition of $\mathcal{L}$ in the following form

$$\mathcal{L} = [\mathcal{L}, \mathcal{L}] \oplus V.$$

Due to the assumption $\dim[\mathcal{L}, \mathcal{L}] \leq n - 2$, we have $\dim V = k \geq 2$. Let $\{e_1, \ldots, e_k\}$ be a basis of $V$.

# 2-local derivations on nilpotent Lie algebras

Let us define a homogeneous non additive function $f$ on $\mathbb{C}^2$ as follows

$$f(y_1, y_2) = \begin{cases} \frac{y_1^2}{y_2}, & \text{if } \ y_2 \neq 0 \\ 0, & \text{if } \ y_2 = 0, \end{cases}$$

where $(y_1, y_2) \in \mathbb{C}$.

According to the assumptions (ii) of the theorem there exists a non zero central element of $\mathcal{L}$. Let us fix it as $z \in Z(\mathcal{L})$. Define an operator $T$ on $\mathcal{L}$ by

$$T(x) = f(\lambda_1, \lambda_2)z, \ \text{ for } \ x = x_1 + \sum_{i=1}^{k} \lambda_i e_i \in \mathcal{L},$$

where $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, k$, $x_1 \in [\mathcal{L}, \mathcal{L}]$. The operator $T$ is not a derivation since it is not linear.

## 2-local derivations on nilpotent Lie algebras

Let us now show that $T$ is a 2-local derivation. Define a linear operator $\delta$ on $\mathcal{L}$ by

$$\delta(x) = (a\lambda_1 + b\lambda_2)z, \quad \text{for} \quad x = x_1 + \sum_{i=1}^{k} \lambda_i e_i \in \mathcal{L},$$

where $a, b \in \mathbb{C}$. Since $\delta|_{[\mathcal{L},\mathcal{L}]} \equiv 0$ and $\delta(\mathcal{L}) \subseteq Z(\mathcal{L})$ the operator $\delta$ is a derivation.

Let $x = x_1 + \sum_{i=1}^{k} \lambda_i e_i$ and $y = y_1 + \sum_{i=1}^{k} \mu_i e_i$ be elements of $\mathcal{L}$. We are going to choose the elements $a$ and $b$ (in the definition of the derivation $\delta$) such that

$$T(x) = \delta(x), \ T(y) = \delta(y).$$

## 2-local derivations on nilpotent Lie algebras

Let us rewrite the above equalities as system of linear equations with respect to unknowns $a, b$ as follows

$$\begin{cases} \lambda_1 a + \lambda_2 b & = f(\lambda_1, \lambda_2) \\ \mu_1 a + \mu_2 b & = f(\mu_1, \mu_2) \end{cases}$$

Since the function $f$ is homogeneous the system has a solution. Therefore, $T$ is a 2-local derivation, as required. $\square$

# 2-local derivations on nilpotent Lie algebras

Let $\mathcal{L}$ be an $n$-dimensional nilpotent Lie algebra with $n \geq 2$. Then $[\mathcal{L}, \mathcal{L}] \neq \mathcal{L}$, otherwise, $\mathcal{L}^k = \mathcal{L} \neq \{0\}$ for all $k \geq 1$.

Suppose that $\dim[\mathcal{L}, \mathcal{L}] = n - 1$. Then direct computations show that

$$\{0\} \neq \mathcal{L}^1 = [\mathcal{L}, \mathcal{L}] = [[\mathcal{L}, \mathcal{L}], \mathcal{L}] = \mathcal{L}^2,$$

and therefore $\mathcal{L}^k = \mathcal{L}^{k+1} \neq \{0\}$ for all $k \in \mathbb{N}$. This contradicts the nilpotency of $\mathcal{L}$. So $\dim[\mathcal{L}, \mathcal{L}] \leq n - 2$.

## 2-local derivations on nilpotent Lie algebras

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**Theorem 2.4.**

Let $\mathcal{L}$ be a finite-dimensional nilpotent Lie algebra with $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

# 2-local derivations on nilpotent Lie algebras

Note that any nilpotent Lie algebra has non trivial center. So, Theorem 2.3 implies the following result.

**Theorem 2.4.**

Let $\mathcal{L}$ be a finite-dimensional nilpotent Lie algebra with $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

The corollary below is an immediate consequence of Theorem 2.4.

# 2-local derivations on nilpotent Lie algebras

Note that any nilpotent Lie algebra has non trivial center. So, Theorem 2.3 implies the following result.

**Theorem 2.4.**

Let $\mathcal{L}$ be a finite-dimensional nilpotent Lie algebra with $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

The corollary below is an immediate consequence of Theorem 2.4.

**Corollary 2.5.**

Let $\mathcal{L}$ be a finite-dimensional abelian Lie algebra with $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

# Outline

# 2-local automorphisms of finite-dimensional semi-simple Lie algebras

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**Conjecture 3.1.**

Let $\mathcal{L}$ be a finite-dimensional semi-simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then every 2-local automorphism of $\mathcal{L}$ is an automorphism.

## 2-local automorphisms of finite-dimensional semi-simple Lie algebras

Concerning 2-local automorphisms on Lie algebras we have the following

**Conjecture 3.1.**

Let $\mathcal{L}$ be a finite-dimensional semi-simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then every 2-local automorphism of $\mathcal{L}$ is an automorphism.

The first step in this direction was made in the paper of Z. Chen and D. Wang
[Z. Chen, D. Wang, 2-local automorphisms of finite-dimensional simple Lie algebras, Linear Algebra Appl. 486 (2015) 335-344.]

# 2-local automorphisms of finite-dimensional semi-simple Lie algebras

Recall that every semi-simple finite-dimensional Lie algebra over $\mathbb{F}$ is a direct sum of simple algebras. The following algebras are simple finite-dimensional Lie algebras:

- $A_n : \mathfrak{sl}_{n+1}(\mathbb{F})$, the special linear Lie algebra;
- $B_n : \mathfrak{so}_{2n+1}(\mathbb{F})$, the odd-dimensional special orthogonal Lie algebra;
- $C_n : \mathfrak{sp}_{2n}(\mathbb{F})$, the symplectic Lie algebra;
- $D_n : \mathfrak{so}_{2n}(\mathbb{F})$, the even-dimensional special orthogonal Lie algebra.

# 2-local automorphisms of finite-dimensional semi-simple Lie algebras

These Lie algebras are numbered so that $n$ is the rank, i.e. the dimension of (all) Cartan subalgebras. These four families, together with five exceptions $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ are in fact the only simple Lie algebras over an algebraically closed field of characteristic zero.

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The main result of the above mentioned paper of Z. Chen and D. Wang is the following

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The main result of the above mentioned paper of Z. Chen and D. Wang is the following

**Theorem 3.2.**

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, $\mathcal{L}$ a finite-dimensional simple Lie algebra of type $A, D, E$ over $\mathbb{F}$. Then every 2-local automorphism $\varphi$ of $\mathcal{L}$ is an automorphism.

# 2-local automorphisms of finite-dimensional semi-simple Lie algebras

## 2-local automorphisms of finite-dimensional semi-simple Lie algebras

The cases of other type simple Lie algebras as well as general finite-dimensional semi-simple Lie algebras remain open.

# Outline

## Local derivations of finite-dimensional semi-simple Lie algebras

The notion of local derivation was introduced by R. Kadison [R.V. Kadison, Local derivations, J. Algebra, 130 (1990) 494-509] and Larson and Sourour [D.R. Larson and A.R. Sourour, Local derivations and local automorphisms of $B(X)$, Proc. Sympos. Pure Math., 51 Part 2, Providence, Rhode Island 1990, pp. 187-194] in the case of Banach and $C^*$-algebras.

# Local derivations of finite-dimensional semi-simple Lie algebras

Kadison proves that each continuous local derivation of a von Neumann algebra is a derivation. This theorem gave rise to studies and several results on local derivations on C*-algebras, culminating with a definitive contribution due to Johnson, [B.E. Johnson, Local derivations on C*-algebras are derivations, Trans. Amer. Math. Soc. 353 (2001) 313-325]
who proved that every (not necessary continuous) local derivation of a C*-algebra is a derivation. A comprehensive survey of recent results concerning local and 2-local derivations on C*- and von Neumann algebras is presented in [Sh. A. Ayupov, K. K. Kudaybergenov, A. M. Peralta, A survey on local and 2-local derivations on C*- and von Neuman algebras, Contemporary Mathematics, AMS, to appear].

# Local derivations of finite-dimensional semi-simple Lie algebras

Here we shall consider in details local derivations on finite-dimensional Lie algebras.
For details we refer to the paper
[Ayupov Sh. A., Kudaybergenov K. K., Local derivations on finite dimensional Lie algebras, Linear Algebra Appl. 493 (2016) 381–398]
First we consider the case of semi-simple Lie algebras.

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**Theorem 4.1.**

Let $\mathcal{L}$ be a finite-dimensional semi-simple Lie algebra over a field of characteristic zero. Then every local derivations on $\mathcal{L}$ is a derivation.

## Outline

# Local derivations of filiform Lie algebras

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Let $\mathcal{L}$ be a finite-dimensional filiform Lie algebra with $\dim \mathcal{L} \geq 3$. Then $\mathcal{L}$ admits a local derivation which is not a derivation.

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The case of general nilpotent finite-dimensional Lie algebras is still an open problem.

# Local automorphisms of finite-dimensional Lie algebras

Examples of pure local automorphisms (i.e. local automorphisms which are not automorphism) on nilpotent matrix algebras were given in [A. P. Elisova, I. N. Zotov, V. M. Levchuk, G. S. Suleimanova, Local automorphisms and local derivations of nilpotent matrix algebras, The Bulletin of Irkutsk State University, Mathematics, 11 (2011) 9–19].

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The problem of description of local automorphisms on finite-dimensional Lie algebras remains open.

THANKS FOR YOUR ATTENTION!