Classification of non-division genetic algebras

Nasir Ganikhodjaev

Department of Computational & Theoretical Sciences
Faculty of Science, International Islamic University Malaysia

Day of Algebra (DoA7)
January 29, 2016

Algebra

An algebra A over a field F is a vector space equipped with a bilinear product. The dimension $m = dim_F A$ of the space A over F is called the dimension of the algebra A over F.

Real Algebra

We will consider real algebras, i.e.,$F = R$. Let $\{a_1, a_2, \cdots, a_m\}$ be a basis of the real vector space A over R. For algebras over a field, the bilinear multiplication from $A \times A$ to A is completely determined by the multiplication of basis elements of A. Conversely, once a basis for A has been chosen, the products of basis elements can be set arbitrarily, and then extended in a unique way to a bilinear operator on A, i.e., so the resulting multiplication satisfies the algebra laws.

Real Algebra

Any algebra can be specified up to isomorphism by giving its dimension (say m), and specifying m^3 structure constants $c_{i,j,k} \in R$. These structure constants determine the multiplication in A via the following rule:

$$a_i a_j = \sum_{k=1}^{m} c_{i,j,k} a_k,$$

The only requirement on the structure coefficients is that, if the dimension m is infinite, then this sum must always converge (in whatever sense is appropriate for the situation).

Genetic Algebra

Below we consider a class of structure coefficients $c_{i,j,k} \in R$ which have a genetic significance and corresponding algebras are called genetic algebras. General genetic algebras are the product of interaction between biology and mathematics. The study of these algebras reveals the algebraic structure of Mendelian and non-Mendelian genetics, which always simplifies and shortens the way to understand the genetic and evolutionary phenomena in real world.

Genetic Algebra

Mathematically, the algebras that arise in genetics are very interesting structures. Many of the algebraic properties of these structures have genetic significance. Indeed, the interplay between the purely mathematical structure and the corresponding genetic properties makes this subject so fascinating.

Genetic Algebra

Recall that a gene is a unit of hereditary information. The genetic code of an organism is carried out on chromosomes. In addition, each gene on a chromosome can take different forms that are called alleles. Suppose we have a random mating population with m distinct alleles. Call them $a_1, a_2, \cdots, a_m$. Let $p_{ij,k}$ be a frequency (probability) that the next generation reproduced by two gametes carrying a_i and a_j will inherit a_k, where $k = 1, 2, \cdots, m$. It is evident that for all $i, j, k = 1, 2, \cdots, m$

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \text{and} \sum_{k=1}^{m} p_{ij,k} = 1.$$

Genetic Algebra

Genetic algebra is an algebra A over the real numbers which has a basis $\{a_1, a_2, \cdots, a_m\}$ and structure constants $p_{ij,k}$, such that a multiplication table is defined as follows:

$$a_i a_j = \sum_{k=1}^{m} p_{ij,k} a_k,$$

where $0 \leq p_{ij,k} \leq 1$ for all i, j, k and

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \text{and} \sum_{k=1}^{m} p_{ij,k} = 1.$$

for $i, j = 1, \cdots, m$.

Genetic Algebra

The algebras that arise in genetics are generally commutative but non-associative. In a general algebra A with genetic realization, an element x in A represents a population if its expression as a linear combination of the basis elements $\{a_1, a_2, \cdots, a_m\}$,

$$x = x_1 a_1 + x_2 a_2 + \cdots + x_m a_m, \tag{1}$$

satisfies $0 \leq x_i \leq 1$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$. Then x_i is the percentage of constituents of type i in population x.

The genetic significance of identity and invertible populations are following: if one of parents belongs to identical population, then second parent and their offspring belong to the same population, and if one of parents belongs to invertible population and also it is known population of their offspring, then one can restore the population of second parent.

The class of genetic algebras is too large to say much about. In M. L. Reed, *Algebraic structure of genetic inheritance*, Bulletin of the American Mathematical Society **34** (1997), 107–130, a brief overview is given of such algebras and proofs of some general properties. For example, it was proven that the algebra with genetic realization is a baric algebra,that is, it admits a non-trivial algebra homomorphism $\omega : A \to R$. The homomorphism ω is called the weight function (or baric function).

Along with genetic algebra we consider on $(m-1)$- dimensional simplex

$$S^{m-1} = \{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in R^m : \text{ for any } i\ x_i \geq 0, \text{ and } \sum_{i=1}^{m} x_i = 1\}. \tag{2}$$

the following mapping $V : S^{m-1} \to S^{m-1}$,

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j \tag{3}$$

with $p_{ij,k} \geq 0$, $p_{ij,k} = p_{ji,k}$ for all i, j, k; and $\sum_{k=1}^{m} p_{ij,k} = 1$, that governs the mathematical model of heredity for large interacting populations of m constituents. Here the numbers x_i represent a fraction of constituents of type i, $i = 1, \cdots, m$ and satisfy the conservation law $\sum_i x_i = 1$.

Volterra qso

The quadratic stochastic operator V is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$. The biological treatment of such operators is rather clear: the offspring repeats one of its parents. It is evident that a qso V is a Volterra if and only if

$$(V\mathbf{x})_k = x_k(1 + \sum_{i=1}^{m} a_{ki}x_i) \tag{4}$$

where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$, $a_{ii} = 0$ and $|a_{ij}| \leq 1$. Here $i, j \in \{1, 2, \cdots, m\}$.

Note that the equations for a Volterra's treatise on the biological struggle for life V. Volterra, *Variations and fluctuations of the number of individuals in animal species living together in Animal Ecology*, Chapman, R.N. (ed), McGraw-Hill, 1931, are distinguished by the form

$$\frac{dx_i}{dt} = x_i \sum_j a_{ij} x_j, \quad i = 1, \cdots, m \tag{5}$$

Here the a_{ij} are biological constants satisfying $a_{ij} = -a_{ji}$, i.e. the $m \times m$ matrix $A = (a_{ij})$ is skew-symmetric. Note that the discrete time system corresponding to dynamical system (5) is defined by the Volterra operator.

Zero-sum games

During the last decades within the game theory, evolutionary and dynamical aspects have exploded (see J.Hofbauer and K.Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge University Press, 1998). Zero-sum games and their evolutionary dynamics were studied by Akin and Losert (E. Akin and V. Losert, *Evolutionary dynamics of zero-sum games.* J. Math. Biology **20** (1984), 231–258). We recall the definition of zero-sum games and show their connection with the Volterra qso. A two-player symmetric game consists of a finite set of strategies indexed by $\Phi = \{1, \cdots, m\}$ and an $m \times m$ payoff matrix (a_{ij}). When an i player meets a j player, their payoffs are a_{ij} and a_{ji}, respectively. Then if $a_{ij} > 0$, we say that a strategy j *is beaten* by a strategy i .In evolutionary game dynamics it is supposed a large population of game players, each with a fixed strategy.

Zero-sum games

The state of the population is a vector in $\mathbb{R}_+^m = \{p \in \mathbb{R}^m : p_i \geq 0\}$, where p_i measures the subpopulation of i strategists. So the total population size is $|p| = \sum_i p_i$. The associated distribution vector $\mathbf{x} = (x_1, \cdots, x_m)$ lies in the simplex S^{m-1}, where $x_i = p_i/|p|$, is the ratio of i strategists to the total population. If a payoff matrix is antisymmetric, then such games are called *zero-sum*, since $a_{ij} + a_{ji} = 0$. If distribution vector $\mathbf{x}' = (x_1', \cdots, x_m')$ is the associated distribution vector in the next moment of time, then as shown by Akin and GGJ, the dynamic is described by Volterra qso (4).
Thus the nonlinear dynamical systems (4) can be reinterpreted in terms of evolutionary games.

Regular QSO

Let V be a qso on S^{m-1}. Assume $\{V^k(\mathbf{x}) \in S^{m-1} : k = 0, 1, 2, \cdots\}$ is a trajectory of the initial point $\mathbf{x} \in S^{m-1}$, where $V^{k+1}(\mathbf{x}) = V(V^k(\mathbf{x}))$ for all $k = 0, 1, 2, \cdots$, with $V^0(\mathbf{x}) = \mathbf{x}$.
A point $\mathbf{a} \in S^{m-1}$ is called a fixed point of a qso V if $V(\mathbf{a}) = \mathbf{a}$.
A qso V is called a regular if for any initial point $\mathbf{x} \in S^{m-1}$, a limit

$$\lim_{n\to\infty} V^n(\mathbf{x}) \tag{6}$$

exists.

Ergodic QSO

In statistical mechanics an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory, an assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time averages may be replaced by space averages. For nonlinear dynamical systems (3) Ulam (S. Ulam, *A collection of mathematical problems*, Interscience Publishers,New-York-London 1960.) suggested an analogue of a measure-theoretic ergodicity, the following ergodic hypothesis:

A nonlinear operator V defined on the unit simplex S^{m-1} is called ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x) \tag{7}$$

exists for any $\mathbf{x} \in S^{m-1}$.

Non-ergodic qso

On the basis of numerical calculations, Ulam conjectured that an ergodic theorem holds for any qso V. In 1977, Zakharevich (M. I. Zakharevich, *On behavior of trajectories and the ergodic hypothesis for quadratic transformations of the simplex*, Russian Math.Surveys **33**, (1978), 265–266) proved that in general this conjecture is false. He considered the following Volterra operator on S^2 :

$$\begin{aligned} x_1' &= x_1(1 + x_2 - x_3) \\ x_2' &= x_2(1 - x_1 + x_3) \\ x_3' &= x_3(1 + x_1 - x_2) \end{aligned} \tag{8}$$

and proved that it is a non-ergodic transformation.

Later in GZ, the authors established a necessary and sufficient condition to be non-ergodic transformation for qso defined on S^2. Generally, the set of Volterra operators is commonly believed to contain all of the pathological types of behavior.

Rock-Paper-Scissors game

Recall that the classical *Rock-Paper-Scissors* is a hand game usually played by two people, where players simultaneously form one of three shapes with an outstretched hand. The "rock" beats scissors, the "scissors" beat paper and the "paper" beats rock. It is described by a Volterra qso (4) (see Akin and Losert). In GGJ the authors proved that a zero-sum game generated by a Volterra operator V be a paper-rock-scissors game if and only the qso V is a non-ergodic transformation.

Qso and corresponding genetic algebras

In a commutative, non-associative algebra, there are several ways to define and interpret the powers of an element. There are two main types of powers which have genetic significance. Let x be an element of a commutative non-associative algebra A. The principal powers are defined to be $x, x^2, x^3, \cdots$, where $x^i = x^{i-1}x$. If A is the genetic algebra and an element P represents a population, then each element P^i of the sequence of principal powers represents a population which resulted from the previous population P^{i-1} mating back with the original population P.

Qso and corresponding genetic algebras

On the other hand, the plenary powers $x, x^{[2]}, x^{[3]}, \cdots$ are defined as $x^{[i]} = x^{[i-1]}x^{[i-1]}$. When P is an element representing a population, the sequence of plenary powers contains the successive generations resulting from random mating within the population, beginning with P. That is, $P^{[2]}$ is the result of the population P mating with itself and $P^{[3]}$ is the result of the population $P^{[2]}$ mating within itself. Both the principal and plenary powers are of biological as well as mathematical results.

Qso and corresponding genetic algebras

Below we will consider Plenary Powers only.

Theorem

Let A be a $m-$dimensional commutative algebra with natural basis $\{a_1, a_2, \cdots, a_m\}$, generated by qso V defined on S^{m-1}. If element $P \in A$ represents a population,i.e., $P = \sum_{k=1}^{m} x_k a_k$ with $x = (x_1, x_2, \cdots, x_m) \in S^{m-1}$, then

$$P^{[n]} = \sum_{k=1}^{m} (V^{n-1}x)_k a_k \tag{9}$$

Qso and corresponding genetic algebras

It is evident that

$$P^{[2]} = \left(\sum_{i=1}^{m} x_i a_i\right)\left(\sum_{j=1}^{m} x_j a_j\right) = \sum_{k=1}^{m}\left(\sum_{i,j=1}^{m} p_{ij,k} x_i x_j\right) a_k = \sum_{k=1}^{m} (V(x)_k a_k$$

and

$$P^{[3]} = \sum_{k=1}^{m}\left(\sum_{i,j=1}^{m} p_{ij,k}(Vx)_i(Vx)_j\right) a_k = \sum_{k=1}^{m} (V(V(x))_k a_k$$

and by induction

$$P^{[n]} = \sum_{k=1}^{m}\left(\sum_{i,j=1}^{m} p_{ij,k}(V^{n-2}x)_i(V^{n-2}x)_j\right) a_k = \sum_{k=1}^{m} (V^{n-1}x)_k a_k$$

Qso and corresponding genetic algebras

Thus this theorem establishes the connection between plenary powers of a population $P = \sum_{k=1}^{m} x_k a_k$ and trajectory of corresponding qso V with initial point $x = (x_1, x_2, \cdots, x_m) \in S^{m-1}$. Therefore it is natural to introduce the following definition.

Definition

A $m-$dimensional commutative algebra A with natural basis $\{a_1, a_2, \cdots, a_m\}$, generated by qso V defined on S^{m-1} is called regular (ergodic) if the corresponding qso V is regular (respectively ergodic).

It is evident that if two $m-$dimensional algebras A_1 and A_2 generated by qso V_1 and V_2 respectively, are isomorphic then both qso V_1 and V_2 are regular or ergodic simultaneously, i.e., the property of algebra to be regular or ergodic is invariant with respect to isomorphism of algebras.

Rock-paper-scissors Algebra

Let us consider Zakharevich's example of Volterra operator defined on S^2:

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 \\ x_2' &= x_2^2 + 2x_2x_3 \\ x_3' &= x_3^2 + 2x_3x_1 \end{aligned} \tag{10}$$

and let A is a $3-$dimensional algebra with natural basis $\{a_1, a_2, a_3\}$ generated by this qso. As noted above, this operator describes a classical paper-rock-scissors game with three strategies. Here a strategy "rock" is beaten by strategy "paper", a strategy "scissors" is beaten by strategy "rock", and strategy "paper" is beaten by strategy "scissors". Let us call the algebra generated by this Volterra operator *Rock-Paper-Scissors Algebra*(*RPS* Algebra).

Rock-paper-scissors Algebra

RPS Algebra is a commutative 3-dimensional algebra with the multiplication table below.

Table: Multiplication Table of the RPS Algebra

	a_1	a_2	a_3
a_1	a_1	a_1	a_3
a_2	a_1	a_2	a_2
a_3	a_3	a_2	a_3

Rock-paper-scissors Algebra

A non-zero element e in an algebra which satisfies the relationship $e^2 = e$ is called an idempotent. In addition to their mathematical importance, idempotents also have genetic significance. If a population P satisfies the equation $P^2 = P$, this means that genetic equilibrium has been achieved after one generation of random mating within the population P.
Mathematically, the existence of an idempotent in an algebra provides a direct sum decomposition of the algebra. Hence, idempotents play a crucial role in describing the general structure. This problem for algebras with genetic realization have been studied by Reed.

Rock-paper-scissors Algebra

There are exactly four idempotents in RPS Algebra, namely $P_1 = a_1, P_2 = a_2, P_3 = a_3$ and $P_4 = 1/3(a_1 + a_2 + a_3)$. The main characteristic of RPS Algebra is the following: for any population P except the idempotents $\{P_i : i = 1, 2, 3, 4\}$ the sequence of averages of plenary powers P does not converge, i.e., the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^{[k]} \tag{11}$$

does not exist. Here a limit is considered with respect to L_1 norm $\|P\| = |x_1| + |x_2| + |x_3|$. Thus RPS Algebra is a non-ergodic genetic algebra.

Example of a Regular Algebra

Let us consider the following Non-Volterra qso defined on S^2

$$\begin{aligned} x_1' &= x_2^2 + 2x_1x_2 \\ x_2' &= x_3^2 + 2x_2x_3 \\ x_3' &= x_1^2 + 2x_3x_1 \end{aligned} \tag{12}$$

and let B is a 3-dimensional algebra with natural basis $\{a_1, a_2, a_3\}$ generated by this qso.

The multiplication table of this algebra has the following form:

Table: Multiplication Table of the Algebra B

	a_1	a_2	a_3
a_1	a_3	a_1	a_3
a_2	a_1	a_1	a_2
a_3	a_3	a_2	a_2

One can see that the pairwise products are the same as above, but the squares are different. In this case there is a single idempotent $P_1 = 1/3(a_1 + a_2 + a_3)$.

The qso (12) have been studied in GR1 and GSJ and proved the following Theorem.

Theorem

The qso (12) is a regular transformation in $IntS^2$, *where* $IntS^2 = \{x = (x_1, x_2, x_3) \in S^2 : x_1 x_2 x_3 > 0\}$.

The main characteristic of the corresponding algebra B is as follows: for any population $P = x_1 a_1 + x_2 a_2 + x_3 a_3$ such that $(x_1, x_2, x_3) \in IntS^2$ the sequence of plenary powers P converges to the idempotent P_1, i.e.,

$$\lim_{n \to \infty} P^{[n]} = P_1. \tag{13}$$

Thus the algebra B is a regular algebra. It is evident that the RPS Algebra and algebra B are not isomorphic.

Real Division Algebra

Recall that an algebra A is called a division algebra if for any element f in A and any non-zero element g in A there exists precisely one element h in A with $f = gh$ and precisely one element d in A such that $f = dg$.
For associative algebras, the definition can be simplified as follows: an associative algebra A is a division algebra iff it has a multiplicative identity element $e \neq 0$ and every non-zero element f has a multiplicative inverse, i.e., an element h with $fh = hf = e$.

Real Division Algebras

The study of real division algebras was initiated by the construction of the quaternion and the octonion algebras in the mid-19th century. In spite of its long history, the problem of classifying all finite dimensional real division algebras is still unsolved.

Outline

1. Genetic Algebras and Corresponding Quadratic Stochastic Operators
2. Non-ergodic Quadratic Stochastic Operators and rock-paper-scissors game
3. Rock-paper-scissors Algebra
4. Non-division Genetic Algebras and their classification
   - Associative Genetic Algebras

In general, the algebras which arise in genetics are commutative but non-associative. Below we consider associative genetic algebras. It is easy to see that an m-dimensional genetic algebra is associative iff for $i, j, k, s = 1, \cdots, m$

$$\sum_{r=1}^{m} p_{ij,r} p_{rk,s} = \sum_{r=1}^{m} p_{ir,s} p_{jk,r}.$$

m=2

Assume $m = 2$. Let A be two-dimensional genetic algebra generated by the following qso:

$$p_{11,1} = z, \;\; p_{12,1} = p_{21,1} = y, \;\; p_{22,1} = x.$$

Theorem

An algebra A is associative iff the coefficients x, y, z satisfy the following equality

$$xz + y - y^2 - x = 0 \tag{14}$$

It is evident that arbitrary $x \in (0, 1], \;\; y \in [x - x^2, 1]$, and

$$z = \frac{x^2 - x + y}{y}$$

satisfy this equality with $y \neq 0$.

Identity

It is easy to show that the element

$$e = \frac{y}{y-x}a_1 + \frac{x}{x-y}a_2 \tag{15}$$

is the multiplicative identity of the algebra A, iff $y \neq x$. Since the coefficients of e have opposite signs, the element e does not represent a population in algebra A.

Inverse

For arbitrary element $f = \alpha a_1 + \beta a_2$ in A the element $h = \gamma a_1 + \delta a_2$ is a multiplicative inverse of f with

$$\gamma = \frac{y(y - xy + y^2)}{(y - x)^2[\alpha(1 - x) - \beta y]}$$

and

$$\delta = \frac{\alpha x(x - 2y + 2xy - x^2 - y^2) - \beta y(2y^2 - 3xy + x^2 - y)}{(y - x)^2(\alpha + \beta)[\alpha(1 - x) - \beta y]}$$

Inverse

Thus element f is invertible if and only if $y - x \neq 0, \alpha + \beta \neq 0$ and $\alpha(1 - x) - \beta y \neq 0$.

Inverse

Let $y - x \neq 0$. Then

$$R_{-1} = \{f = \alpha a_1 + \beta a_2 \in A : \alpha + \beta = 0\}$$

and

$$R_{\frac{1-x}{y}} = \{f = \alpha a_1 + \beta a_2 \in A : \alpha(1 - x) - \beta y \neq 0\}$$

are one-dimensional linear subspaces of all non-invertible elements in A. It is evident that there exists unique non-invertible population

$$f = \frac{y}{1 + y - x} a_1 + \frac{1 - x}{1 + y - x} a_2$$

inverse

Thus if $x = y$, the algebra A does not have multiplicative identity and respectively all elements of algebra A are non-invertible. Such algebra is called *fully non-division algebra*.

If $x \neq y$, then there are two one-dimensional linear subspaces of non-invertible elements in A. Such algebra is called *partially non-division* algebra.

Fully non-division two dimensional genetic algebras

Let $x = y \neq 0$. Then from (2) we have $z = x$. Thus fully non-invertible two-dimensional associative algebra A with standard basis a1, a2 is determined by qso

$$p_{11,1} = x, \quad p_{12,1} = p_{21,1} = x, \quad p_{22,1} = x.$$

Let A_x be fully non-division two dimensional genetic algebras

Theorem

Two associative fully non-division two dimensional genetic algebras A_x and $A_{x'}$ are isomorphic iff $x = x'$.

Partially non-division two-dimensional algebra

Let $A_{x,y}$ be a partially non-division two-dimensional genetic algebra with $x \neq y$ and $y \neq 0$.

Theorem

Two partially non-division associative two-dimensional genetic algebras $A_{x,y}$ and $A_{x',y'}$ are isomorphic iff $x = x'$ and $y = y'$.

The End of The Presentation

Thanks for your attention !!!